RATIONAL EXPRESSIONS OF ARC LENGTH OF PLANE CURVES BY TANGENT OF MULTIPLE ARC AND CURVES OF DIRECTION - Part XV -

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Abstract

We present some examples of parametrized plane curves which have at times a computable arc length in the context of integration in finite terms, by elementary algebraic expressions and their logarithms. We present polynomials and rational arc lengths which are important classes of curves useful in CAGD and PH-curves.

1 Computable arc lengths of plane curves

In books on plane curves we note that arc length are rarely expressed by polynomials or rational formulas since $\sqrt{dx^2 + dy^2}$ is rarely a simple integrable expression. Among conics - algebraic of order 2 - for only two very specific examples with computable arc length : circle and parabola arc lengths can be expressed by elementary functions (Logarithms, exponentials and trigonometric circular / Hyperbolic and their inverses) see (2). The general case requires to create new classes as elliptic functions and others. A long time ago Liouville and Serret studied classes of curves with same arc length as the one of the ellipse. Serret gave an infinite class of such curves in polar coordinates. We use another easier way and look for curves with an arc length expression that can occasionnally be computed only by elementary functions. In a paper of NAM 1922 M. Weill (1) gives several examples of such curves and different methods to generate from a constraint on the parametric equations - $[x(t), y(t)]$ in an orthonormal coordinate system. The element of arc is given by the standard integral :

$$s - s_0 = \int_0^t \sqrt{dx^2 + dy^2} = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.dt$$

A difficulty is the presence of the radical that leads in general to a non computable integral as in the case of the ellipse and lots of well known curves. So we look for special cases where the ds^2 is a perfect square and sometimes leads to easier computations. A well known example is $dx = (1-t^2)dt$ and $dy = 2t.dt$ so $ds = (1+t^2)$ and $s_{(o-t)} = t + \frac{t^3}{3}$ and the curve is the Tschirnhausen's cubic. This can be modified and give a wider expression :

$$u = (1-t^2).f(t) \qquad v = 2.t.f(t)$$

M. Weill generalizes in (1) this last case using formula : $u^2 + v^2 = z^{2m}$ and $u + iv = (\alpha + \imath\beta)^{2m}$ and he obtains :

$$u = \alpha^{2m} - C^2_{2m}\alpha^{2m-2}\beta^2 + ...$$
$$v = C^1_{2m}\alpha^{2m-1}\beta - C^3_{2m}\alpha^{2m-3}\beta^3 + ...$$
$$u = Re[(1+\imath t)^{2m}] \qquad v = Im[(1+\imath t)^{2m}]$$

1.1 Arc length defined from $\cos n\theta$ and $\sin n\theta$

The preceeding example comes from a special case of the expression of tangent functions of multiple arc when n=2 : $\tan 2\gamma = \frac{2.t}{1-t^2}$. We suppose n is an integer and set $z = \alpha + \imath\beta$, $\tan\theta = \beta/\alpha$ and $\rho = \sqrt{\alpha^2+\beta^2}$.

$$z = \rho e^{\imath\theta} = \rho.(\cos\theta + \imath\sin\theta)$$

Following J.S. Calcut in (3) we can extand the method to calculate tangents of multiple angles using de Moivre/Euler formula, we have :

$$\tan\theta = \frac{1}{\imath}\frac{e^{\imath\theta} - e^{-\imath\theta}}{e^{\imath\theta} + e^{-\imath\theta}} = \frac{1}{\imath}\frac{e^{2\imath\theta} - 1}{e^{2\imath\theta} + 1}$$
$$\arctan t = \frac{1}{2\imath}\ln\frac{1+\imath t}{1-\imath t}$$
$$F_n(t) = \tan n\theta = \frac{A_n(t)}{B_n(t)} = \tan\frac{1}{2\imath}\ln\left(\frac{1+\imath t}{1-\imath t}\right)^n$$

So we get :

$$\boxed{A_n(t) = \frac{1}{2\imath}[(1+\imath t)^n - (1-\imath t)^n]}$$
$$\boxed{B_n(t) = \frac{1}{2}[(1+\imath t)^n + (1-\imath t)^n]}$$

And finally :

$$\boxed{A^2_n(t) + B^2_n(t) = (1+t^2)^n}$$

1.2 Table of function A(t) and B(t) : $A_n^2(t) + B_n^2(t) = (1+t^2)^n$

The following table gives the first values of $A_n(t)$ and $B_n(t)$:

n	$A(t)$	$B(t)$
0	0	1
1	t	1
2	$2t$	$1-t^2$
3	$3t-t^3$	$1-3t^2$
4	$4t-4t^3$	$1-6t^2+t^4$
5	$5t-10t^3+t^5$	$1-10t^2+5t^4$
6	$6t-20t^3+6t^5$	$1-15t^2+15t^4-t^6$
7	$7t-35t^3+21t^5-t^7$	$1-21t^2+35t^4-7t^6$
8	$8t-56t^3+56t^5-8t^7$	$1-28t^2+70t^4-28t^6+t^8$
9	$9t-84t^3+126t^5-36t^7+t^9$	$1-36t^2+126t^4-84t^6+9t^8$
10	$10t-120t^3+252t^5-120t^7+10t^9$	$1-45t^2+210t^4-210t^6+45t^8-t^{10}$

1.3 A recursion formula for $A_n(t)$ and $B_n(t)$ and a composition law

Functions $A_n(t)$ and $B_n(t)$ can be computed by the following recursion from J.S. Calcut in (3):

$$\boxed{A_{n+1}(t) = t.B_n(t) + A_n(t) \quad B_{n+1}(t) = B_n(t) - t.A_n(t)}$$

We have :

$$F_n(t) = \tan n\theta = \frac{A_n(t)}{B_n(t)}$$

and we can combine formulas for two values of the index, say n and m then, when F exists - see (3) :

$$F_{nm}(t) = F_n \circ F_m(t) = F_m \circ F_n = F_{mn}(t)$$

This permutable composition law is useful to compute expressions of F(t), A(t) or B(t). Garry J. Tee in paper (4) studies different permutable polynomials and rational functions among them the tangent function.

1.4 Computable arc length derived from tangent of multiple arc

Expressions in the table above can be used to find algebraic curves with rational arc length or even polynomial arc length. These are algebraic identities verifying the pythagorean property. So if $x'(t) = B(t)$ and $y'(t) = A(t)$ then $x = \int B_n(t)dt$ and $y = \int A_n(t)dt$ give parametric equation of such a curve.

n=0 vertical line : x=t, y=a.
n=1 Parabola : $x = t, \quad y = t^2/2$
n=2 Tschirnhausen's cubic : $x = t - t^3/3, \quad y = t^2$
n= 3 curve : $x = t - t^3, \quad y = 3t^2/2 - t^4/4$
n= 4 curve : $x = t - 2t^3 + t^5/5, \quad y = 2t^2 - t^4$
n= 5 curve : $x = t - 10t^3/3 + t^5, \quad y = 5t^2/2 - 5t^4/2 + t^6/6$
n= 6 curve : $x = t - 5t^3 + 3t^5 - t^7/7, \quad y = 3t^2 - 5t^4 + t^6$
....
The arc lengths of these curves are : $s = \int (1+t^2)^{n/2} dt$.

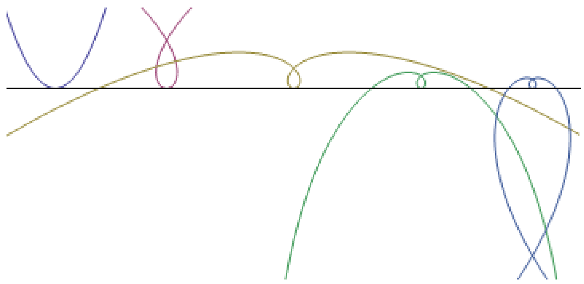


Figure 1: The five first curves of the sequence

If n is even n=2p then the arc length is a polynomial in t as $(1+t^2)^p$ is one. If n is odd (n=2p+1) we can put either $t = \tan u, \quad du = dt/(1+t^2)$ or $t = \sinh u, \quad dt = \cosh u.du$ and then $s = \int (1/\cos u)^{n+1} du$ or $s = \int (\cosh u)^{n+1} du$
We can also use $A_n^2(t) + B_n^2(t) = (1+t^2)^n$ with n even (=2p) to find unicursal curves with rational arc length.
For $k \leq p$ arc length is computable by elementary functions :

$$dx = \frac{A_n(t)}{(1+t^2)^k}.dt \qquad dy = \frac{B_n(t)}{(1+t^2)^k}.dt$$

$$s = \int ds = \int \sqrt{dx^2 + dy^2} = \int (1+t^2)^{(n/2)-k}.dt$$

$$s = \int ds = \int \frac{1}{(\cos u)^{n-2k+2}}.du$$

Since above pythagorean expressions :

$$K^2(t).A_n^2(t) + K^2(t).B_n^2(t) = K^2(t).(1+t^2)^n$$

- with K(t) an arbitrary rational function of t - are algebraic identities, we can replace t by any function of a new variable and obtain new curves with sometimes a computable arc length.

So the general equations for computable arc length are :

$$\boxed{x(t) = \int K(t).A_n(t).dt \qquad y(t) = \int K(t).B_n(t).dt}$$

And the expression of the arclength is :

$$\boxed{s(t) = \int K(t).(1+t^2)^{n/2}.dt}$$

The formulas for A(t) and B(t) in the above table give a generalization of the well known identity :

$$\boxed{[a(t)^2 - b(t)^2]^2 + [2.a(t).b(t)]^2 = [a(t)^2 + b(t)^2]^2}$$

We put $t = \frac{a}{b}(b \neq 0)$ in the formulas for A(t) and B(t) and get homogeneous polynomial multiplying by b^{2n} and new pythagorean identities when n is even or $(a^2 + b^2)$ is a square. We drop parameter (t) to get simpler fomulas:
For n=3 :

$$\boxed{[3ab^2 - a^3]^2 + [-3.a^2b + b^3)]^2 = [a^2 + b^2]^3}$$

For n=4 :

$$\boxed{[4ab^3 - 4a^3b]^2 + [b^4 - 6a^2b^2 + a^4)]^2 = [a^2 + b^2]^4}$$

For n=5 :

$$\boxed{[5ab^4 - 10a^3b^2 + a^5]^2 + [b^5 - 10a^2b^3 + 5a^4b]^2 = [a^2 + b^2]^5}$$

For n=6 :

$$\boxed{[6ab^5 - 20a^3b^3 + 6a^5b]^2 + [b^6 - 15a^2b^4 + 15a^4b^2 - a^6)]^2 = [a^2 + b^2]^6}$$

These expressions have interesting permutable properties linked to common factors of n. For example if we replace in : $[a^2 - b^2]^2 + [2.a.b]^2 = [a^2 + b^2]^2$ (n=2) value a by $2ab$ and value b by $(b^2 - a^2)$ we get the identity for n=4 because n=2.2.

1.5 Arc length defined from $\cosh n.u$ and $\sinh n.u$

Hyperbolic functions give analog formulas without using $\imath$ as for the real de Moivre functions :

$$\boxed{(\cosh u + \sinh u)^n = \cosh n.u + \sinh n.u = e^{n.u}}$$

$$\boxed{(\cosh u - \sinh u)^n = \cosh n.u - \sinh n.u = e^{-n.u}}$$

The computations for circular tangents can be expanded with a similar method to calculate hyperbolic tangents of multiple arguments using de Moivre/Euler formula, we have :

$$\alpha = \tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^{2u} - 1}{e^{2u} + 1}$$

$$u = \operatorname{arctanh} \alpha = \frac{1}{2} \ln \frac{1+\alpha}{1-\alpha}$$

$$G_n(\alpha) = \tanh n\alpha = \frac{C_n(\alpha)}{D_n(\alpha)} = \tanh \frac{1}{2} \ln \left(\frac{1+\alpha}{1-\alpha}\right)^n$$

We have :

$$\boxed{C_n(\alpha) = \tfrac{1}{2}[(1+\alpha)^n - (1-\alpha)^n]}$$

$$\boxed{D_n(\alpha) = \tfrac{1}{2}[(1+\alpha)^n + (1-\alpha)^n]}$$

And the pythagorean identity function of n, for $\alpha \leq 1$:

$$\boxed{D_n^2(\alpha) = C_n^2(\alpha) + (1-\alpha^2)^n}$$

Or in a second case, if $\alpha \geq 1$:

$$\boxed{C_n^2(\alpha) = D_n^2(\alpha) + (\alpha^2-1)^n}$$

1.6 Table of functions $C(\alpha)$ and $D(\alpha)$: $D_n^2(\alpha) - C_n^2(\alpha) = (1-\alpha^2)^n$

The following table gives the first values of $C_n(\alpha)$ and $D_n(\alpha)$, only for $\alpha \leq 1$:

n	$C(\alpha)$	$D(\alpha)$
0	0	1
1	α	1
2	2α	$1+\alpha^2$
3	$3\alpha+\alpha^3$	$1+3\alpha^2$
4	$4\alpha+4\alpha^3$	$1+6\alpha^2+\alpha^4$
5	$5\alpha+10\alpha^3+\alpha^5$	$1+10\alpha^2+5\alpha^4$
6	$6\alpha+20\alpha^3+6\alpha^5$	$1+15\alpha^2+15\alpha^4+\alpha^6$
7	$7\alpha+35\alpha^3+21\alpha^5+\alpha^7$	$1+21\alpha^2+35\alpha^4+7\alpha^6$
8	$8\alpha+56\alpha^3+56\alpha^5-8\alpha^7$	$1+28\alpha^2+70\alpha^4+28\alpha^6+\alpha^8$
9	$9\alpha+84\alpha^3+126\alpha^5+36\alpha^7+\alpha^9$	$1+36\alpha^2+126\alpha^4+84\alpha^6+9\alpha^8$
10	$10\alpha+120\alpha^3+252\alpha^5+120\alpha^7+10\alpha^9$	$1+45\alpha^2+210\alpha^4+210\alpha^6+45\alpha^8+\alpha^{10}$

1.7 A recursion formula for $C_n(\alpha)$ and $D_n(\alpha)$ and a composition law

Functions $C_n(\alpha)$ and $D_n(\alpha)$ can be computed by the following recursion :

$$\boxed{C_{n+1}(\alpha) = \alpha.D_n(\alpha) + C_n(\alpha) \qquad D_{n+1}(\alpha) = D_n(\alpha) + \alpha.C_n(\alpha)}$$

We have :

$$G_n(t) = \tanh n\theta = \frac{C_n(\alpha)}{D_n(\alpha)}$$

and we can combine formulas for two values of the index, say n and m then, when G exists :

$$G_{nm}(\alpha) = G_n \circ G_m(\alpha) = G_m \circ G_n = G_{mn}(\alpha)$$

All this can be done just setting $t = \imath\alpha$ since the computations are similar.

1.8 Computable arc length derived from hyperbolic tangent of multiple arc

As in the case of ArcTangent expressions in the table above can be used to find algebraic curves with rational or polynomial arc length. These are algebraic identities verifying the pythagorean property. Here we make the assumption that $|\alpha| \leq 1$, so if $x'(\alpha) = C_n(\alpha)$ and $y'(\alpha) = (1-\alpha^2)^{n/2}$ then if $|y| \leq 1$: $x = \int C_n(\alpha)d\alpha$ and $y = \int (1-\alpha^2)^{n/2}d\alpha$ give parametric equations of such a curve.
We have also if $|y| \geq 1$: $x = \int C_n(\alpha)d\alpha$ and $y = \int (\alpha^2-1)^{n/2}d\alpha$ give parametric equations of such a curve. We set $\alpha = \tanh u/2$ (or eventually : $\alpha = 1/\tanh u/2$. In the list below we suppose $|\alpha| \leq 1$:

n= 1 curve : $x = \alpha^2/2, \quad y = \int (1-\alpha^2)^{1/2}d\alpha$
n=2 Tschirnhausen's cubic : $x = \alpha^2, \quad y = \alpha - \alpha^3/3$
n= 3 curve : $x = 3\alpha^2/2 + \alpha^4/4, \quad y = \int (1-\alpha^2)^{3/2}d\alpha$
n= 4 curve : $x = 2\alpha^2 + \alpha^4, \quad y = \alpha - 2\alpha^3/3 + \alpha^5/5$
n= 6 curve : $x = 3\alpha^2 + 5\alpha^4 + \alpha^6, \quad y = \alpha - \alpha^3 + 3\alpha^5/5 - \alpha^7/7$
n= 8 curve : $x = 4\alpha^2 + 14\alpha^4 + 28\alpha^6/3 + \alpha^8, \quad y = \alpha - 4\alpha^3/3 + 6\alpha^5/5 - 4\alpha^7/7 + \alpha^9/9$
n= 10 curve : $x = 5\alpha^2 + 30\alpha^4 + 42\alpha^6 + 15\alpha^8 + \alpha^{10}, \quad y = \alpha - 5\alpha^3/3 + 2\alpha^5 - 10\alpha^7/7 + 5\alpha^9/9 - \alpha^{11}/11$

The arc lengths of these curves are : $s = \int D_n(\alpha)d\alpha$.
If n is even n=2p as $D_n(\alpha)$ is a polynomial then the arc length is also a polynomial in α .

We can also use $D_n^2(\alpha) = C_n^2(\alpha) + (1-\alpha^2)^n$ with n an integer to find unicursal curves with rational arc length ($\alpha < 1$).
For $k \leq n$ arc length is computable by elementary functions :

$$dx = \frac{C_n(\alpha)}{(1-\alpha^2)^k}.d\alpha \qquad dy = \frac{(1-\alpha^2)^n}{(1-\alpha^2)^k}.d\alpha$$

$$s = \int ds = \int \sqrt{dx^2 + dy^2} = \int D_n(\alpha).\frac{1}{(1-\alpha^2)^k}.d\alpha$$

Since above pythagorean expressions :

$$K^2(\alpha).C_n^2(\alpha) + K^2(\alpha).(1-\alpha^2)^n(\alpha) = K^2(\alpha).D_n(\alpha)$$

- with $K(\alpha)$ an arbitrary rational function of α - are algebraic pythagorean identities.

So the general equations for computable arc length are :

$$\boxed{x(\alpha) = \int K(\alpha).C_n(\alpha).d\alpha \qquad y(\alpha) = \int K\alpha).(1-\alpha^2)^n(\alpha).d\alpha}$$

And the expression of the arclength is :

$$\boxed{s(\alpha) = \int K(\alpha).D_n(\alpha).d\alpha}$$

The formulas for C(α) and D(α) in the hyperbolic table give as for the tangent function pythagorean identities of the form :

$$C_n^2(\alpha) + (1-\alpha^2)^n = D_n^2(\alpha)$$

Or if $\alpha \geq 1$:

$$D_n^2(\alpha) + (\alpha^2 - 1)^n = C_n^2(\alpha)$$

We put $\alpha = \frac{a}{b} (b \neq 0)$ in the formulas for C(α) and D(α) and get homogeneous polynomial multiplying by b^{2n} and new pythagorean identities. Functions $a(\alpha)$ and $b(\alpha)$ give formulas in the same way as for the tangent function D_n:

For n=2 :

$$\boxed{[2.a(\alpha).b(\alpha)]^2 + [b(\alpha)^2 - a(\alpha)^2]^2 = [a(\alpha)^2 + b(\alpha)^2]^2}$$

For n=3 :

$$\boxed{[3ab^2 + a^3]^2 + [b^2 - a^2]^3 = [b^3 + 3.a^2b]^2}$$

For n=4 :

$$\boxed{[4ab^3 + 4a^3b]^2 + [b^2 - a^2]^4 = [b^4 + 6.a^2b^2 + a^4)]^2}$$

2 Computable arc lengths from plane caustics

G. Humbert showed at the end of nineteenth century that plane curves with a rational arc length are the caustics by reflection of algebraic plane curves for parallele light rays. This gives a method to find curves with computable arc length (see Parts XIII-XIV).

2.1 Computable arc lengths from plane caustics of $y = y(x)$

In paper (5) R. Allaro and S. Althoen give a formula for a computable arc length of classes of plane curves. The curve is given in explicit form $y = f(x)$:

$$s = \int_{xo}^{x1} \sqrt{f'^2(x) + 1}.dx \qquad x_0 \leq x \leq x_1$$

The integral is simpler if the quantity under the radical $\sqrt{f'^2(x) + 1}$ is a perfect square. Assume $f'^2(x)+1 = s^2(x)$ so $(s(x)+f'(x))(s(x)-f'(x)) = 1$. If we set a new function $g(x) = s(x) + f'(x)$ then $\frac{1}{g(x)} = s(x) - f'x)$ and :

$$y = f(x) = \frac{1}{2}\int \Big(g(x) - \frac{1}{g(x)}\Big).dx$$

Let $g(x)$be any function, these formulae give classes of functions with computable arc length for example when g(x) is a rational function of x.
If we put $g(x) = x^n$ then we get pursuit curves - see Part VII -:

$$y = f(x) = \frac{1}{2}\int \Big(x^n - \frac{1}{x^n}\Big).dx$$

$$y = \frac{1}{2}\Big[\frac{1}{n+1}.x^{n+1} + \frac{1}{n-1}.x^{1-n}\Big]$$

When n=1 by a direct computation :

$$y = f(x) = \frac{1}{2}\int \Big(x - \frac{1}{x}\Big).dx$$

$$y = \frac{1}{2}\Big[\frac{x^2}{2} - \log x\Big] \qquad \text{the special pursuit curve.}$$

If we put $g(x) = \exp x$ then we get the catenary- see Part VI -:

$$f(x) = \frac{1}{2}\int \Big(e^x - e^{-x}\Big).dx$$

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

This process to create curves with a computable arc length can be interpreted in geometric terms as the search for curves of direction. They are defined as the caustics by reflection of algebraic plane curves for parallele light rays. Examples are curves of pursuit when algebraic.

2.2 Computable arc lengths from plane caustics of parametric curve $x = x(t)$ and $y = y(x)$

The above method to generate a curve with a computable arc length can be generalized to parametric curves $x = x(t)$ and $y = y(t)$, t a real parameter : We set the derivatives $y'(t) = f(t)$ and $x'(t) = g(t)$ then :

$$s = \int_{to}^{t1} \sqrt{f(t)^2 + g(t)^2}.dt \qquad t_0 \leq x \leq t_1$$

The condition to get a simpler integral is that the function under the radical $\sqrt{f^2(t) + g^2(t)}$ is a perfect square. So the three functions s'(t), f(t) and g(t) must form a triple of pythagorean functions (pythagorean hodograph).
Assume $s'^2(t) = f^2(t) + g^2(t)$ so $(s'(t) + f(t))(s'(t) - f(t)) = g^2(t)$. We use a new arbitrary function h(t) and if we set $g(t).h(t) = s'(t) + f(t)$ then $\frac{g(t)}{h(t)} = s'(t) - f(t)$ and if :

$$x(t) = \int g(t).dt$$

and if :

$$y(t) = \int f(t) = \frac{1}{2} \int g(t).\Big[h(t) - \frac{1}{h(t)}\Big].dt$$

then :

$$s_{to-t}(t) = \frac{1}{2} \int_{to}^{t} g(t).\Big[h(t) + \frac{1}{h(t)}\Big].dt$$

Let $g(t)$ and $h(t)$ be any function, these can some times produce classes of functions with computable arc length for example when g(t) and h(t) are rational functions of t and the integral is computable. If the function h(t) is a tangent $h(t) = \tan u$ then the tangent to the resulting curve has for slope:

$$\tan V = \frac{dx}{dy} = \frac{2.h(t)}{h^2(t) - 1} = -\tan 2u$$

The angle V is the angle between negative y-axis (parallele to light rays) and oriented tangent at current point M or M' on the two associated curves by the OTT.
This can be geometrically interpreted if this tangent is the line MM' in the construction of the caustic of light rays coming from y direction and reflected by the initial plane curve with parametric coordinates. The construction is given by the orthogonal tangent transformation (OTT) (see Part XIII) :

$$\tan V_M = -h(t)$$

and symetrically :

$$\tan V_{M'} = \frac{1}{h(t)}$$

$$\tan V_M . \tan V_{M'} = -1$$

as required for the transformation (orthogonal tangents). And this is the general construction - with help of two arbitrary functions $[g(t),\ h(t)]$ - of plane caustics :

$$\boxed{x(t) = \int g(t).dt \qquad y(t) = \frac{1}{2}\int g(t).\left[h(t) - \frac{1}{h(t)}\right].dt}$$

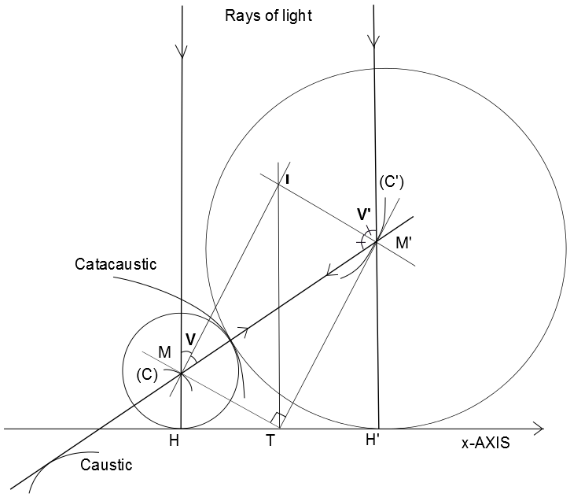


Figure 2: Arbitrary function defines a couple of OTT associated curves

2.3 Recall of orthogonal tangent transformation (OTT)

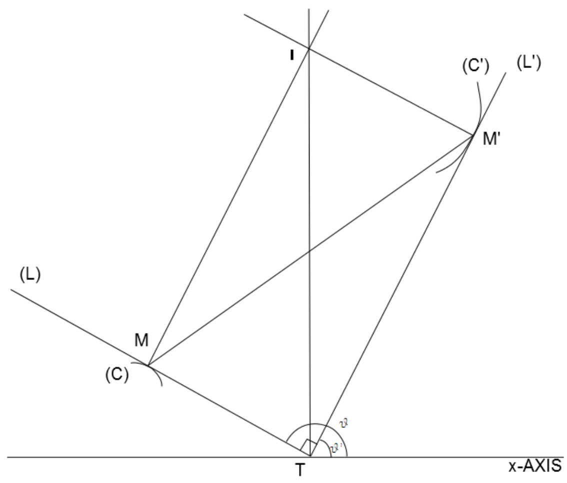
A curve (C) is given by its parametric coordinates in an orthonormal system [x(t), y(t)]. The tangent at current point cuts x-axis at T : the normal in T to this tangent envelope a new curve (C'). The following equations give the coordinates [X(t), Y(t)] of the new curve (C') envelope of the tangents mapped by the orthogonal tangents transformation (OTT) :

$$\boxed{X(t) = x(t) - 2.[\frac{x'(t)}{y'(t)}].y(t) = x(t) - 2.\frac{y(t)}{\tan V_M}}$$

$$\boxed{Y(t) = [\frac{x'(t)}{y'(t)}]^2.y(t) = \frac{y(t)}{(\tan V_M)^2}}$$

There is a relation for sub-tangent HT of the two curves (M) and (M') :

$$\overline{HT} = -\overline{H'T} = \frac{y(t)}{\tan V_M} = -\frac{Y(t)}{\tan V_{M'}}$$



Orthogonal tangents transformation (OTT)

Figure 3: Orthogonal tangents transformation : L $\longmapsto$ L'

These formulas give a general form of caustics equations in the plane. Since the two arbitrary functions g(t) and h(t) are necessary, we give in the following section somme examples of cautics.
We use a parameter t as the function h(t)=t because, the tangent or the cotangent of the couple of initial curves asociated by OTT. This simplifies the problem and gives a geometric interpretation of the function f(t). So the general equations of this subset are :

$$\boxed{x(t) = \int g(t).dt \qquad y(t) = \int g(t).\left[\frac{t^2-1}{2t}\right].dt}$$

This shows the double angle of caustic transformation ($t = \tan u$) :

$$\boxed{\tan V = \frac{dx}{dy} = \frac{2t}{t^2-1} = \tan(-2u)}$$

2.4 Extension of the formula of double arc to multiple arc

The above formula for parametric equations of a family of curves of direction using two arbitrary functions can be adapted to multiple tangent formulas (circular and hyperbolic) given above. We keep g(t) and just modify the formula by replacing h(t), considered as a tangent function, by the expressions of $tan(n.\theta)$ and this gives new formulas for curves of direction :

$$x(t) = \int g(t).dt \qquad y(t) = \int \frac{1}{2}.g(t).\left[\frac{A_n(t)}{B_n(t)} - \frac{B_n(t)}{A_n(t)}\right].dt$$

This is an elementary method to find parametric equations of curves of direction with computable arc length. The expression of the element of arc is the following :

$$s(t) = \int \frac{1}{2}.g(t).\left[\frac{A_n(t)}{B_n(t)} + \frac{B_n(t)}{A_n(t)}\right].dt$$

There is a similar version of these formulas for the Hyperbolic functions, replacing A, B and $1 + t^2$ by hyperbolic expressions with C, D and $\pm(1 - \alpha^2)$.These produces as the tangent function hyperbolic pythagorean expressions in a same way (see part VII).

3 Rolling variable circles generating caustics of given curve for parallele light rays

In paper (6) the author shows how to use rolling circles to generate the caustics of a plane curve. These variable circles are the critical circles at current point of the mirror curve. The focal point after reflection lies on these circles. Their radius is $\frac{R_c}{4}$ and they are in the general case different at each current point of the mirror curve and have a second envelope. A point "angulary fixed" on the variable critical circle rolling on this second envelope generates the caustic. By angulary fixed we mean a cumulative angle which can be computed by an integral on the elementary rotation of the circle perfectly determined $\int_{so}^{s1} \frac{4.ds}{R_c}$.
The critical circles have for envelope two curves : the first given mirror curve A and another curve B and can be considered as a variable circles rolling on the curve B. A point angulary fixed on this critical circle will generate the caustic of curve A.

3.1 Nephroid caustic of a circle for parallele ligth rays

It is a special case since mirror has constant radius R so critical circle has radius : R/4. The second envelope B is a concentric circle (to the mirror circle) of radius R/2 and we have the well known generation. The curve A is a circle of radius R, the curve B is a concentric circle of constant half radius so the caustic is the well known nephroid.

3.2 Astroid : caustic of a deltoid for parallele ligth rays

A second example in (7) is the special example of the caustic of the deltoid : it is an equal astroid for light rays coming from any direction. The critical circles (r=$R_c/4$) are the ones centered on the cusps circle and tangent to the deltoid. And we know (Part XIV) that a point angulary fixed on the this variable circle rolling on the deltoid decribes the caustic an astroid. The curve A is the deltoid and the curve B is a 3-epicycloid with common cusps. The caustic is generated by variable circles above rolling on the 3-epicycloid. It is an Astroid.

3.3 Tschirnhausen's cubic : caustic of a parabola for parallele ligth rays

A third special example is the caustic of the parabola which is a Tschirnhausen's cubic (TC) for light rays coming from any direction. But this time dimension of the caustic is different for each direction since the TC are transformed by a similarity centred at the common focus of TC and Parabola (see part IV). The critical circle passes at the focus of the parabola. The curve A is the parabola, curve B is reduced to the focus. The caustic, a Tschirnhausen's cubic (TC), is generated by a point angulary fixed to the variable circle passing ('rolling') through the focus and tangent to the parabola. All these TC generated by light ray coming parallely from all possible directions, are the same TC transformed by a similarity centered at the focus of the parabola. Parabola and TC share the same unique focus.

3.4 The general case : caustic of a closed curve for parallele ligth rays

The general case presents no such regularity as in the three above cases and the caustics are all different and depend on the direction of the parallele light rays and Boyle's generation (with focal point on the critical circle of radius $= R_c/4$ at intersection with reflected ray at current point) is a simple and useful method. The resulting caustic is also a closed curve. But it remains to find the geometric relations between the two curves like for the rotation index, etc.

And a question remains to explain if the three special cases above for which the resulting caustic is a unique equal curve as nephroid and astroid or similar as Tschirnhausen's cubic are the only one or if there are others.

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Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
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