LOGARITHMIC SPIRAL ABERRANCY OF PLANE CURVES AND CONICS - Part XVI -

C. Masurel

17/05/2016

Abstract

A local property of a plane curve is examined as a generalization of approximation of a curve by its tangent line or its osculating circle (the curvature) which is symetrical w.r.t. the normal. The aberrancy is a local parameter measuring the shift from this symmetry. Abel Transon in his 1841 paper write "define the aberrancy by mean of the osculating logarithmic spiral could not present any difficulty" but used conics instead of the logarihmic spiral which however gives a simpler purely geometric concept of a "deviation".

1 The Logarithmic Spiral.

Descartes was the first to define, in a letter to Mersenne of 1638, the logarithmic (equiangular) spiral (LogSpi or LS) by its fondamental property as the curve that cuts at the same angle V a set of convergent lines in a point called the pole. The curve was defined by this simple geometric property. The spiral is not algebraic but its rectification was know very soon by Torricelli (1647) and though a transcendental equation it is in fact an elementary curve generalizing naturally the circle and the line.

2 Fondamental property of the Logarithmic Spiral.

The equiangular spiral is the solution of the isogonal trajectory for the lines passing through a fixed point in the plane. The common angle V between vector radius and tangent at current point is the only necessary parameter since the vector radius of all these spirals has the same range $]0, \infty[$ for polar angle $\in]-\infty, +\infty[$. For this curve a rotation around the pole S is equivalent

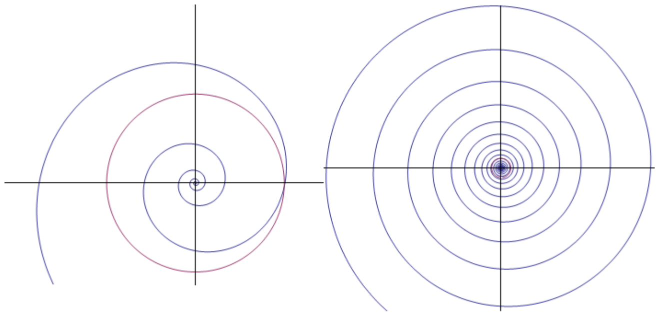

Figure 1: Examples of Logarithmic spirals

to a scaling or homothetie from S. Two spirals with same V and same pole can be superposed by a rotation around S or a scaling from S. In the rest of this paper S is the pole of an osculating logarithmic spiral and O the center of an osculating conic (O is at ∞ for the parbola).
The polar equation (ρ, θ) results from the fact that V is constant since :

$$\boxed{\rho\frac{d\theta}{d\rho} = \tan V \longmapsto \frac{d\rho}{\rho} = \frac{d\theta}{\tan V}}$$

$$\boxed{\rho = e^{k(\theta-\theta_o)} = ae^{k\theta} \leftrightarrow k = 1/\tan V \text{and } a = e^{k\theta_o}}$$

We recall two classical properties of LogSpi, if C_1 is the first center of curvature at a current point M then :
1- The circle with diameter MC_1 passes through the pole S.
2- The projection of the center of curvature C_1 on the corresponding vector radius of successive evolutes is the common pole S. So the second center of curvature is on the vector line MS.

3 Circle - line - Logarithmic spirals

The LS has two special important cases :
1- if angle $V = \pm\pi/2$ the LS becomes a circle with center S,
2- if angle $V = 0$ or π the LS is a half-line ending at the pole S.
This shows that the LS has many properties in common with the line and the circle despite its disturbing strangeness as a transcendental plane curve.

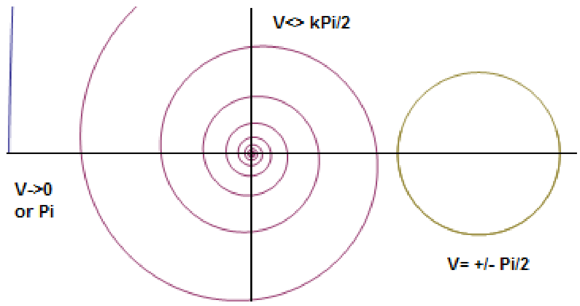

Figure 2: Three cases of LogSpi : half-line, general LS and circle

4 Definition and history of the aberrancy of a plane curve (L. Carnot - A. Transon)

The definition of aberrancy, as a local property of a plane curve, by Carnot in (1) of 1803 is equivalent to :
"The axis of aberrancy at a point M is the limiting position of the line MI with I the midpoint of a secant line parallele to the tangent as the secant approaches the tangent". Abel Transon resumes the idea in a paper (2) of 1841 in NAM that seems to be the real beginning of the concept. The center of aberrancy is the limiting intersection of two nearby axis. Later G. Salmon translated the french word "deviation" by the term "aberrancy" and the name is conserved until now but it is not exactly the meaning of the initial idea. It is a parameter that measures the deformation w.r.t. the osculating circle by mean of an osculating conic. The geometric definition of Carnot is adapted for conics but not easy to use for a general curve in the plane.

4.1 Aberrancy with conics.

A. Transon in his paper of 1841 studies the concept of aberrancy using conics. A property of conics has modeled the concept. All types of conic have a diameter, a line on which moves the middle of a chord parallel to a fixed direction (for conics it is more than a local property: it is global). Two diameters of a conic section are said to be conjugate if each chord parallel to one diameter is bisected by the other diameter. So a parabola, or even any conic that osculates a general curve at a point M, gives by the position of its center O the aberrancy direction MO, and angle δ. O is at infinity for the parabola. This gives the definition of "conical" aberrancy. Note that all osculating conics give the same aberrancy axis for the osculated curve.

4.2 Aberrancy with a parabola

Between the summit and the point at infinity along the parabola the radius of curvature varies from p to ∞ and the angle of aberrancy from 0 to $\pi/2$ at ∞. The axis of aberrancy is the parallele to the axis of osculating parabola. Vaziri (5) and Schot (6) give the relation defining the radius of aberrancy (also called "oblique curvature") at the current point :

$$\boxed{R_{ab} = \frac{\rho \cos \delta}{1 - \frac{d\delta}{d\theta}}}$$

There is a different sign before δ' in their formulas because they use complementary angles V and $\pi/2 - V$.
Note that this formula derived in (6) does'nt use the definition of Carnot. Since for conics δ is not constant the construction of the center of (conical) aberrancy is not the projection of the center of curvature on the aberrancy axis. This should lead to the research of more appropriate curves with a constant aberrancy.

5 Cesaro curves

In paper (3) on curvature of conics E. Cesaro gives the definition of his general curves by a formula for a local property of his curves :

$$\boxed{\frac{d\rho}{ds} = m.\frac{\alpha}{\beta}}$$

Coordinates (α, β) are the ones of a point fixed in the plane of osculating cuve. ρ is the radius of curvature at M, s the arc length and m a parmeter. The ratio $\frac{\alpha}{\beta}$ is the tangent of the angle of conical aberrancy and it cuts the second radius of curvature on the opposite side at $\frac{1}{3}$ of C_1C_2.
In case of conics this point is the center O of the osculating conic and the parameter $m = -3$. This formula is equivalent to the geometric fact that the line MO cuts the second radius of curvature at third on the opposite side of the normal. The sign of Cesaro's parameter m has been changed here. Any osculating conic defines a unique conical aberrancy angle.
Cesaro's curves include important classical curves when m=1 : Cycloidals, Ribaucour curves and sinusoidal spirals. In this special case Cesaro defines a director circle and he gives a new curious property :
"The center of curvature at current point M is the intersection of the normal with the polar of M w.r.t. the director circle" which is often used as the definition of Cesaro's curves.
A limiting case of sinusoidal spirals $\rho^n = \sin n\theta$ for $n \to 0$ is the Logarithmic spiral.

6 Back to the origin of deviation : 1841

A. Transon in (2) p 195 offers to use the logarithmic spiral to define the "deviation". He writes :
"The LogSpi has a constant deviation and the reflected ray at M (on the curve) is the axis of deviation supposing the vector radius is the light ray. This curve plays, with repect to the second modification of curvature similar to the circle for the first one. The circle has constant curvature, the logarithmic spiral has constant aberrancy (deviation in french)." He adds :
"Determining the osculating logarithmic spiral at any point presents no difficulty." This is the easy track we follow in the present paper.
Later (1913) E. Turriere gave in (4) a definition of the same idea with the osculating LogSpi, indirectly, because his paper doesn't deal with the aberrancy.

6.1 Osculation by a logarithmic spiral

E. Turriere gave the method to construct the osculating logarithmic spiral using the two first radii of curvature of the given curve (C) at M. This is enough to find the pole S :
- Trace the circle of diameter MC_1, C_1 is the first center of curvature,
- Place C_2, the second center of curvature of (C),
- Draw the the line MC_2 : it cuts the preceeding circle at S.
S is the pole of the osculating LS. We only need to know the first and second centers of curvature of (C) at M.

6.2 An important property of the Logarithmic spiral : the constant "deviation"

The aberrancy of a logarithmic spiral should be computed using the above geometric definition of Carnot. But this is not really convenient nor accessible so we follow another path and define directly by the LogSpi the notion of "deviation". We leave the word aberrancy for osculating conics.
The Logspi has an essential local auto similarity property each portion seen from S in a same angle is similar to any other part in the same conditions. And a LogSpi defines a constant angle between the normal at a point and the reflected vector radius. This angle is the "Transon deviation" of the LogSpi. The center of deviation is in the same way the symmetric of S w.r.t. the normal at M. Axis and center of deviation are unique and easy to construct by these definitions.
And in the case of LogSpi the deviation $\delta = \pi/2 - V$ is constant as V is along the curve.

6.3 Osculating logarithmic spiral and osculating circle

For a given curve the pole S of the osculating LS at point M is on the circle of diameter MC. In the general case pole S can be found anywhere on this circle.
There are two special cases :
- when S is at C then the point M is a vertex and deviation angle is 0 and the LS is the classical curvature circle at M.
- S is at M and angle of deviation is 0 or π then the tangent half-line has 3 points of contact : it is an inflection point and the LS is the inflection half-tangent.
This is an additional argument to confirm interest to use an osculating LogSpi for defining deviation : it generalizes the curvature in a natural way.

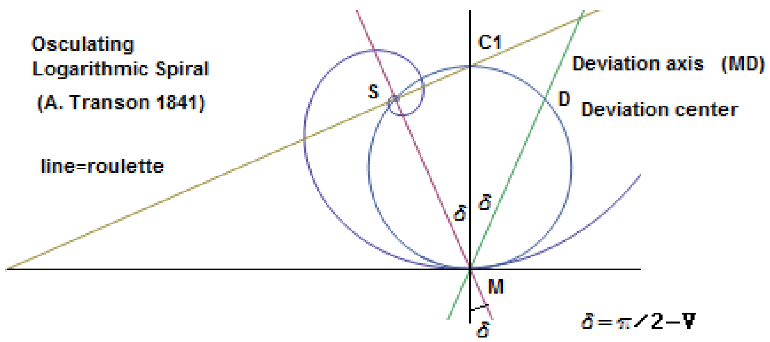

Figure 3: Osculation of a curve with a logarithmic spiral

6.4 Relation between (conical) aberrancy and (LogSpi) "deviation"

In a paper (8) of 1912 about anharmonic curves (or self-projective or W-Kurven in German) S.W. Reaves gives details about this connection and shows that "the osculating ellipse at all points of the logarithmic spiral $\rho = e^{m\theta}$ remains similar to itself" as a consequence of self similarity of the LogSpi. The center O of the osculating ellipse is on the circle constructed on the radius of curvature MC_1 as the diameter. Pole S of the osculating SL is also on this circle.
The results of Reaves' paper reflect the proportionality between tangents :

$$\boxed{\tan \delta_{conic} = \frac{1}{3} \tan \delta_{LogSpi}}$$

$$\boxed{\tan\delta_{conic} = \frac{1}{3}\frac{\rho'}{\rho} \text{ and } \tan\delta_{LogSpi} = \frac{\rho'}{\rho}}$$

In paper (9) K. Maeda gives, among many, two theorems about osculating figures in the plane in relation with aberrancy :

A - The locus of the pole of any logarithmic spiral having a contact of at least the second order with a curve at a point M is a circle of diameter MC tangent at M. This gives the deviation.

B - The locus of the focus of any parabola having contact of at least second order with a curve at point M is a circle tangent at M of diameter $\frac{1}{2}MC$. This is important to study caustics by reflection.

7 Gregory's transformation and the logarithmic spiral

The LS is often presented as a transcendental curve and algebraic curves are supposed to be easier to study. But it is in fact a simple curve with internal symmetry that brings the curve near the circle : It is a circle with an angle V and fascinating properties.

For example a classical one : if we draw all tangents to the spiral $\rho = e^{k\theta}$

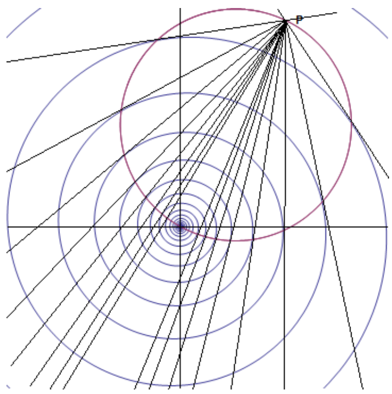

Figure 4: Property of tangents to LogSpi from a point P

from a fixed point P. From all tangent points the line OP is seen to subtend the fixed angle $\pi - \arctan k$ so all these points of the tangents are situated on a circle passing through the pole of the spiral (13) p212.

V.E. Adler in a paper (10) of 2009 on tangential maps gives this other property :

"Theorem 1. Consider the intersection points of the logarithmic spiral C with one of its tangent. The tangents through these points meet on the same spiral".

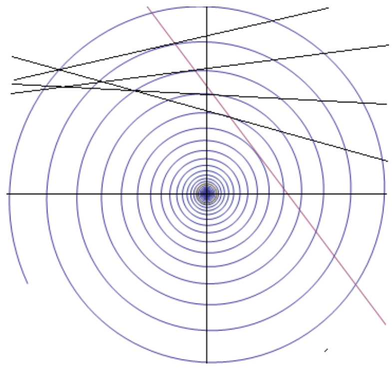

Figure 5: Property of LogSpi quoted by V.E. Adler

In Part I on Gregory's transformation we used the theorem of Steiner-Habich and a chain of successive pedals of an initial curve to create couples of ground/wheel. The LogSpi is a special example because the pedal is the same LogSpi - "eadem mutata resurgo" - so all curves are a unique logarithmic spiral with same angle V. Note that the ground-line is a half-line only the part above the xx' axis.
The roulette of the pole of a LS is a half-line and the wheel corresponding to the same half-line (for base line axis xx') is the same spiral. All the successives wheels or rolling curves are the same LS and the ground is a half-line at angle $\pi/2 - V$ with xx'. Since there is only one LS for one value of V.

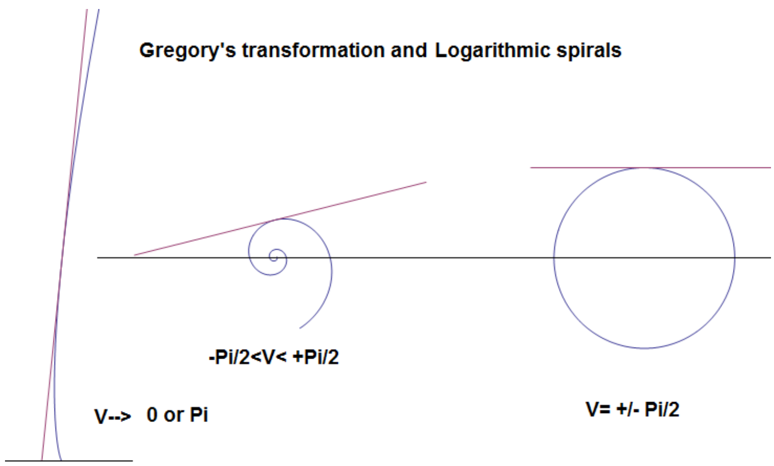

Figure 6: 3 wheels for vertical, inclined and horizontal lines

8 Second envelope of a variable circle tangent to a curve

A classical configuration studied by E. Cesaro, E. Turriere, R. Goormaghtigh is the envelope of a variable circle tangent to a given curve (C). In general for smooth curves (C) this envelope is the the curve (C) and another one. The two curves are linked by the variable circle. E. Turriere in paper (4) shows that the point of contact with the second envelope is the pole of the LogSpi osculating (C) at current point M.

Cesaro had proved in (12) p71 that the second envelope (called by him cusp track) when curve (C) is a LS and the circles are the family of osculating circles of the LS, focuses in a unique fixed point : the pole of the LS. For a general curve the locus of the poles of osculating LS is the second envelope when the circle is on the side of the curvature center.

These circles and their envelopes are a kind of generalisation of cycloidals curves generated by moving or rolling circles inside a couple of concentric circles. All this shows that there is an important geometric relation between osculating LS and envelopes of families of variable circles tangent to a given curve.

Goormaghtigh has studied some properties of these families of circles and their envelopes in 1916.

And these families of circles, specially when the two envelopes are closed curves, crossing each other or not, seem to present interesting geometric properties by rolling on one or the other envelope curves.

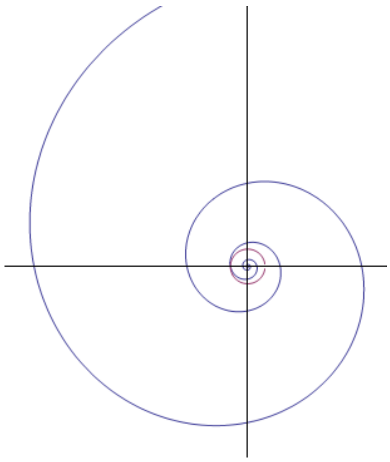

Figure 7: A Logarithmic Spiral

References :

(1) Geometrie de position p 475-476 - L.N.M. Carnot (1803)

(2) Sur la courbure des lignes et des surfaces. Abel Transon J.M.P.A. (Mai

1841)
(3) Sur la courbure des coniques E Cesaro NAM 1886.
(4) Sur les spirales logarithmiques osculatrices une courbe plane. E.Turriere. L'enseignement des mathematiques 15 (1913)
(5) Sur quelques courbes lies au mouvement d'une courbe plane dans son plan. Ahmad-Vaziri Abolghassem. These (1938) supervised by E. Turriere.
(6) Aberrancy : Geometry of the third derivative Steven H. Schot (1978)
(7) Geometrical properties of the penosculating conics of a plane curve. Steven H. Schot (1979)
(8) On the projective differential geometry of plane anharmonic curves - Samuel W. Reaves (1912)
(9) On some osculating figures of the plane curve. Kazuhiko Maeda - Sendai (1942)
(10) The tangential map and associated integrable equations - V.E. Adler (2009)
(11) Descartes Logarithmic spiral - Chantal Randour.
(12) Sur les lignes de poursuite. E. Cesaro - NAM fevrier 1886.
(13)Advanced plane Geometry - C.Zwikker (1950) - North-Holland Publishing Company, Amsterdam.
A - Nouvelles annales de mathematiques (1842-1927) Archives Gallica
B - Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
This article is the XVI^{th} on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.

Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr