CESARO'S CURVES
A GENERALIZATION OF CYCLOIDALS

- Part XVII (draft 1)-

C. Masurel

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Abstract

In the years 1886-1900 E. Cesaro published in Mathesis and the Nouvelles Annales de Mathematiques several papers about a large class of curves with interesting properties. These are expansion of a special property of conics and cycloidals. Later he studied a class defined by an intrinsic equation with two parameters λ and μ and a length R as subclasses of his curves. After a short story of the Cesaro curves, we present a generalization of the cycloidals using Mc Laurin and pedal transformations.

1 Successive evolutes of a curve

Evolutes and involutes of plane curves go back to the work of huygens but was really explained by Jakob Bernoulli with help of new calculus in Leibniz style about 1690. Radius of curvature was expressed by the first and second derivatives. Later geometers used the successive radii of curvature. In this domain the circle is very special : its first radius is $\neq 0$ the others are equal to 0. In the opposite direction the successive involutes of a point present some interesting properties studied by J.J. Sylvester about 1868 under the name of "Cyclodes" : they have a finite number of radii different from zero like the involute of the circle. He mentions the spiral of Norwich which is a special case of involute of the involute of a circle and is a curve with the property of sturmian spirals $\rho = k.R_c$ when k=1, the pole is at the center of the circle. If two curves have in common some of the first radii of curvature they are osculating curves. The circle of curvature is the simplest case. The next case are curves for which the two first radii coincide. These curves have same R and same R_1 so same LogSpi deviation (see Part 16).

2 Intrinsic coordinates of Cesaro :

In paper [3] Cesaro gives the solution of a question Nr 461 of Mennesson in Mathesis 1886 Tome V p192 : "The second envelope of cicumferences with diameter the radii of curvature of a cycloidal is the inverse of this cycloidal". This leads him directly to the general equations of cycloidals in an intrinsic coordinate frame (s, R) :

$$R^2 = as^2 + 2bs + c \qquad (1)$$

If $a < 0$ these are cycloidals associated to ellipses, and for a=0 involute of the circle. If $b^2 - ac = 0$ we get the logarithmic spiral. If $a > 0$ we have the pseudo cycloidal associated to each of the two kinds of hyperbolas. At that time, as mentioned at the end of the paper he had convinced himself that intrinsic methods were useful beside the other techniques of the differential calculus and he intended "to gather them and coordinate them in a didactic work of intrinsic geometry".
And Cesaro published a few years later a book [2] on these methods to study plane curves by intrinsic equation f(R, s)=0 where R is the radius of curvature and s the arc length counted from a given origin on the curve.

3 New curves generalizing conics in 1888

In [5] using an analogy with a property of conics Cesaro makes an hypothesis about his curves using the second radius of curvature.
First he gives for conics the following relation :

$$\frac{dR}{ds} = -3\frac{\alpha}{\beta} \qquad (2)$$

where R is the first curvature radius, s the arc length at current point and (α, β) is a fixed point in the plane of the curve that he called the pole O. For the conics this pole is the center of the conic, at ∞ on the axis for the parabola. For conics $CC_1 = -3CP$ if P is the intersection point of OM and second radius of curvature. I use a different orientation from Cesaro's : C to C_1 as positive direction.
Then Cesaro, replacing -3 by a new parameter m, in the previous formula generalizes to a "very large class of curves" with equation, :

$$\frac{dR}{ds} = m\frac{\alpha}{\beta} \qquad (3)$$

If C is center of curvature at M then $CP = mCC_1$. In the same issue Cesaro comes back to the suject and begins with this definition of his curves.

4 On two classes of remarkable plane curves 1888

The beginning of this paper [6] is : "We shall study plane curves, provided with a pole O, such that the second radius of curvature is cut in a constant proportion by the vector ray between the current point and the pole".
This leads him to an equation :

$$\alpha x + \beta y = (n+1)\beta\rho = u^2 - R^2 \qquad (4)$$

α and β are the coordinates of the pole in the local frame (origin M tangent and normal). The value n is linked to m by : $m = \frac{n-1}{n+1}$.
The value R^2 is a constant of integration interpreted as the square of the radius of the fixed circle. Equation (4) is the one of the polar of M w.r.t. the fixed circle (O, R) and Cesaro uses it as a new definition given as "equivalent" to the previous one : the radius of curvature is proportional (n+1) to the segment of normal between M and the polar of this point w.r.t. the directrix circle. Cesaro's curves include as special cases : cycloidals and conics. For the cycloidals the value of (n+1) is 1, for the conics it is -1 so the symmetric of the circle diameter MC w.r.t. the tangent is orthogonal to the fixed circle which is for conics the Monge orthoptic circle with same center as the conic. Note that using Cesaro's parameter n, the value 0 gives the cycloidals and -2 gives the conics so :

$$\text{Cycloidals : } \alpha x + \beta y = \beta\rho \qquad\qquad \text{Conics : } \alpha x + \beta y = -\beta\rho$$

For conics the pole O is the center, for cycloidals it is the center of the fixed circle orthogonal to all the circles constructed on radius of curvature MC as the diameter. There is a difference between the two classes since for conics it is not the circle with diameter MC but the symetrical of this circle w.r.t. the tangent at the current point M. For conics the fixed circle is the Monge or orthoptic circle of the central conic.

Cesaro deduces from these calculations the following intrinsic equation (s, ρ) for his curves :

$$s = \frac{n+1}{n-1} \int \frac{d\rho}{\sqrt{\frac{R^2}{c^2}(\frac{\rho}{c})^{2\frac{(n+1)}{(n-1)}} + (n+1)(\frac{\rho}{c})^{\frac{2n}{n-1}} - 1}} \qquad (5)$$

This equation includes the special cases of conics for n=-2 and cycloidals for n=0.

5 On two classes of remarkable plane curves 1888 (continued).

The two definitions imply for a Cesaro curve a real number n and a fixed point O(α , β) (the center or the pole) in the plane of the curve. Cesaro

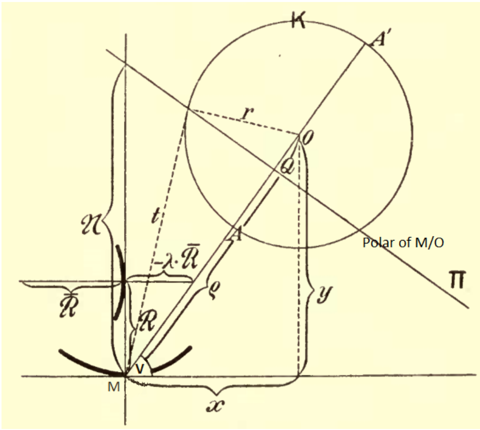

Figure 1: Definition of Cesaro curves (from Wieleitner 1908). (x, y) are (α, β) in the text.

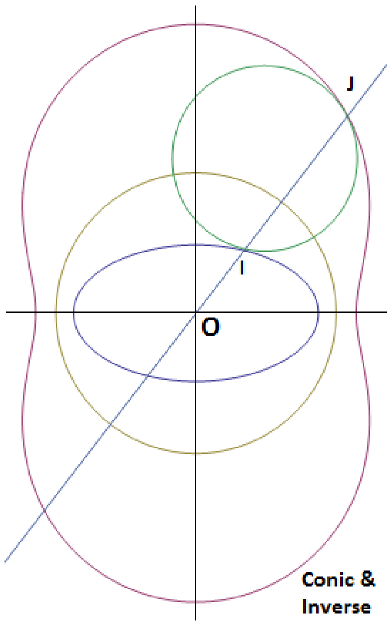

Figure 2: Property of symmetric w.r.t. to the current tangent of osculating circle for conics : it is orthogonal to orthoptic or Monge circle.

presents exemples of his curves but they are, if we move away the circle, in the following classes : cycloidals, sinusoidal spirals, Ribaucour curves, and conics. So the problem becomes to find other curves out of these four known classes respecting the conditions of Cesaro : plane curves with a pole O and a parameter n and such that OM cuts the second radius of curvature in a constant ratio.

In the same paper Cesaro explains that, since R radius of fixed circle is part

of the definition of his curves there is, for a given n, one in the class of Ribaucour curves ($R = \infty$) and one in the class of sinusoidal spiral ($R = 0$). Cesaro curves are a larger class including and generalizing these two classes and also conics and cycloidals.
We recall the second definition : "curves such that the radius of curvature is proportional (k) to the segment of normal between M and the polar of this point w.r.t. the directrix circle". So the circle constructed on MC as diameter is not orthogonal to the fixed circle (O,R), but this circle, dilated in a constant ratio from M (current point of Cesaro's curve), is orthogonal to this fixed circle (note that for the special case of cycloidals the proportion is equal to one and the circle is orthogonal). He uses this fact of dilated circle in paper [8]-p490 mentioning 3 special cases : 1-dilated circles pass through a fixed point (sinusoidal spirals), 2-dilated circles are orthogonal to a fixed line (Ribaucour curves) and 3-dilated circles are tangent to a fixed line : this gives an interesting class of curves (see Part III and VI) including nephroid, Poleni's curve or Tschirnhausen's cubic among many curves as involutes of caustics by reflection for parallele light rays (see Part XII and XIII).

6 Remarks on osculation (NAM mars 1890)

In this paper [7] Cesaro gives many cases of special curves by mean of two parameters λ and μ and uses the parabola and right hyperbola as fondamental curves to derive the others. It is not clear what was really the aim of these speculations. It could be the continuing search for the generalization of his class of curves. He makes the link between the degree of osculation or the number of points coinciding at current point M. For tangent line : 2 points, osculator circle : 3 points, if osculating curve has the two first radii of curvature, for example if we use a logarithmic spiral with pole at N, in common then there are 4 coinciding points at M and so on. This reduces osculation by a given Cesaro curve with a pole to a geometric problem. He calculates the relations between a few first radii of curvature to define some well know curves (most often cycloidals and conics) and try to find equations of the two special cases which for any value of m belong to Ribaucour curves and sinusoidal spiral. For n=-2 Conics : Parabola and rectangular hyperbola, for n= 0 : Cycloidals : Cycloid and Logarithmic spiral. But he gives no example for new cases for n $\neq 0, -2$.
In the same paper Cesaro gives the intrinsic equation (s, ρ) with examples among cycloidals, Ribaucour curves and sinusoidal spirals :

$$s = \int \frac{d\rho}{\sqrt{\frac{\lambda}{\mu}\left[(\frac{\rho}{a})^{2\mu} - 1\right]}} \qquad (6)$$

We note that the intrinsic equation of conics is not of this type and that the general differential intrinsic equation of Cesaro's curves (5) is not really easy to integrate.

7 On a class of remarkable plane curves NAM 1900

Cesaro comes back to his curves in [8] and studies the curves defined by an intrinsic equation (s, ρ) which gives essentially cycloidals, Ribaucour curves and sinusoidal spirals :

$$s = \int \frac{\lambda d\rho}{\sqrt{(\frac{\rho}{a})^{\mu} - 1}} \qquad (7)$$

This equation is not so general as the one (5) given for definition in his paper of 1888 where he presents his curves.
Two special values of λ and μ define an individual curve. A relation between these parameters defines a subclass as $\pm\lambda = \mu - 1$: Ribaucour curves, or $\pm 2\lambda = \mu$: sinusoidal spirals. The subclass defined by $\pm\lambda = \mu$ are curves such that osculator circle dilated in a constant proportion from the current point are tangent to a fixed line.

8 On some envelopes (Leopold Braude NAM 1913)

L. Braude in the paper of 1913 [9] examines directly by means of Cesaro's methods the envelope of a line passing through the current point M and cutting the second radius of curvature at P in the proportion λ (used by him in place of the above m) and tries to find the curves such that the envelope of the line MP_λ is a fixed point in the plane of the curves. For the cycloidals ($\lambda = 1$) this envelope is a point (pole), the center of the fixed circle, for central conics ($\lambda = -3$) it is the center (at ∞ for the parabola).
Note that L. Braude restricts the definition of Cesaro's curves to the values of λ which correspond to the existence of a pole.
The two known classes are in fact a little miracle in the set of Cesaro's curves and finding new classes is not easy since intrinsic differential equation seems to be in general not integrable by elementary functions. This could explain the fact that Cesaro gives only cycloidal or conical examples.
L. Braude finds cycloidals for λ=1 and a class characterized by a complicated equation U=0 (proved by Cesaro in [2] including the four first radii of curvature of the curve) a kind of differential formula - where U is a relation between the four first radii of curvature of the searched curves. Since U=0 annulates the radius of curvature of the envelope which is reduced to a fixed

point, the pole O. But he doesn't give anymore information about eventual new classes.

$$U = \frac{(\lambda-1)R_1[\lambda^2R^2 + R_1^2 + 3\lambda(RR_2 - R_1^2) + \lambda^2R(R_1R_2 - RR_3)]}{[\lambda^2R^2 + R_1^2(1-\lambda) + \lambda RR_2]^2}$$

We have the evident solution $\lambda = 1$: cycloidals, $R_1 = 0$: circles but the others are the solutions of the last factor of the numerator :

$$[\lambda^2R^2 + R_1^2 + 3\lambda(RR_2 - R_1^2) + \lambda^2R(R_1R_2 - RR_3)] = 0$$

We know that for conics the parameter λ is -3. These values lead to the two classes of curves that can be expressed by elementary functions cycloidals and conics.
In paper [7] of 1890 Cesaro gives the characteristic equation of conics with the first four radii of curvature :

$$40R_1^3 - 36R^2R_1 + 9R^2R_3 - 45RR_1R_2 = 0$$

But it seems not really easy to find, comparing the two equations, the curves corresponding to U=0 for the cancellation of the last factor above. The form of the intrinsic equations of the two classes confirms the fact that cycloidals and conics are different among Cesaro's curves.

$$s - s_0 = \frac{1}{3}\int \frac{d\rho}{\sqrt{\left[1-(\frac{a\rho}{b^2})^{\frac{2}{3}}\right]\left[(\frac{b\rho}{a^2})^{\frac{2}{3}}-1\right]}} \qquad \text{(conics)}$$

Furthermore cycloidals expression of $\tan V$ is a true tangent (see below) for cycloidals, sinusoidal spirals and Ribaucour curves but not for the central conics.

9 On a class of remarkable plane curves (Goormaghtigh NAM 1919)

Goormaghtigh has considered the same class of curves defined with two parameters ν and λ analog to the intrinsic equation already given by Cesaro (1890). He uses same equation as Cesaro in 1900 paper but with modified parameters:

$$s = \int \frac{\nu d\rho}{\sqrt{(\frac{\rho}{a})^{-2\nu\lambda} - 1}} \qquad (8)$$

And gives intrinsic equation of the evolutes of these curves (barocentric curves):

$$\rho = \frac{s}{\nu}\sqrt{(\frac{\rho}{a})^{-2\lambda\nu} - 1}$$

For barocentric curves the abcisse is proportional to a power of the arc length. This paper which resumes and echoes the one of Cesaro 1900 draws up the list of the three special cases of osculating circles dilated in a constant proportion subdued on condition : 1- passes through a fixed point (sinusoidal spirals), 2- is orthogonal to a fixed line (Ribaucour curves) or 3- is tangent to a fixed line (involute of caustics by reflection Ribaucour curves for parallele rays).

10 Euler polar tangential equation of Cesaro's curves (L. Braude 1917)

In paper [10] L. Braude has given the equation in the Euler form as the envelope of a line in the plane defined by :

$$x.\cos\theta + y.\sin\theta - p(\theta) = 0$$

For Cesaro's curves $p(\theta)$ is given by the following equation (the λ here = above $\lambda - 1$):

$$\theta = \int \frac{dp}{\sqrt{\frac{p^{\lambda+2}}{a^{\lambda}} - p^2 + b^2}} \qquad (9)$$

The equation of the pedal is (replacing p by r and θ by ω) :

$$\omega = \int \frac{dr}{\sqrt{\frac{r^{\lambda+2}}{a^{\lambda}} - r^2 + b^2}}$$

And the equation of the ground (xx'= base line) associated with this pedal is :

$$x = \int \frac{y.dy}{\sqrt{\frac{y^{\lambda+2}}{a^{\lambda}} - y^2 + b^2}}$$

L. Braude gives also the expression of the radius of curvature for Cesaro's curves :

$$R = p + \frac{d^2p}{dp^2} = \frac{\lambda+2}{2.a^{\lambda}}p^{\lambda+1}$$

For $\lambda = 0$ corresponding to cycloidals L. Braude gives : $p'^2 - p.p" = a^2$ an by integration he finds :

$$p = \frac{a}{n}\cos n(\theta - \theta_o) \qquad \text{or} \qquad p = a.\theta$$

The cycloidals are antipedal of the rhodoneas.
For $\lambda = -4$ corresponding to conics, equation (9) gives :

$$\theta = \int \frac{dp}{\sqrt{\frac{a^4}{p^2} - p^2 + b^2}}$$

$$\theta = \int \frac{pdp}{\sqrt{a^4 + b^2p^2 - p^4}}$$

The expression under radical $a^4 + b^2p^2 - p^4 = (d - p^2)(e - p^2) = 0$ has two roots :

$$d, e = \frac{b^2 \pm \sqrt{b^4 + 4a^4}}{2}$$

$$\theta = \frac{1}{2} \int \frac{2p.dp}{\sqrt{(d - p^2)(e + p^2)}} = \frac{1}{2} \int \frac{2du}{\sqrt{(d - u)(e + u)}}$$

After integration :

$$\theta = \arctan \sqrt{\frac{d - u}{e + u}} \longrightarrow \tan^2 \theta = \frac{d - u}{e + u}$$

We set $\tan \theta = t$ then :

$$u = \frac{d - e.t^2}{1 + t^2} = d \cos^2 \theta - e. \sin^2 \theta = p^2$$

This is the well known equation of central pedal of conics as expected.

11 Central parametric equations of cycloidals in polar coordinates (Gomez Teixeira)

In Tome 3 of [11] Gomez Teixeira gives parametric equations in polar coordinates of cycloidals when the pole is at the center of director circle. ρ is the vector radius and θ the polar angle and a parameter u. R and r are the radii of fixed director and rolling circles. These equations are not so popular as the cartesian x-y standard ones but they allow to show closeness of cycloidals, sinusoidal spirals and Ribaucour curves.

$$\boxed{\left(\frac{\rho}{R}\right)^2 = \frac{u^2+1}{u^2+m^2} \qquad m = \frac{R}{R+2r}} \qquad (11)$$

$$\boxed{\theta = \frac{1}{m} \arctan \frac{u}{m} - \arctan u} \qquad (12)$$

The parameter u is :

$$\boxed{u = \sqrt{\frac{m^2\rho^2 - R^2}{R^2 - \rho^2}}} \qquad (13)$$

$$d\theta = \frac{(\rho^2 - R^2)^{(n+1)/2}}{\rho\sqrt{a^{2n}\rho^2 - (\rho^2 - R^2)^{n+1}}} d\rho$$

The polar differential equation of Cesaro curves given by Gomez Teixeira is:

$$\boxed{[\rho^2 - R^2]^{n+1} = a^{2n}\rho^2 \sin^2 V} \qquad (14)$$

So $\sin V$ is a rational function of ρ and V depends only of the distance from O. The property used by Gomez-Teixeira is the second definition of Cesaro : the radius of curvature at the current point is a constant proportion of the segment MC between M and the intersection of the normal with the polar of M w.r.t. the fixed director circle (O, R). Note that inflexions and cusps of the curves are situated only on this circle. For cycloidals (n=-2) circle constructed on MC as diameter is orthogonal to the fixed circle.

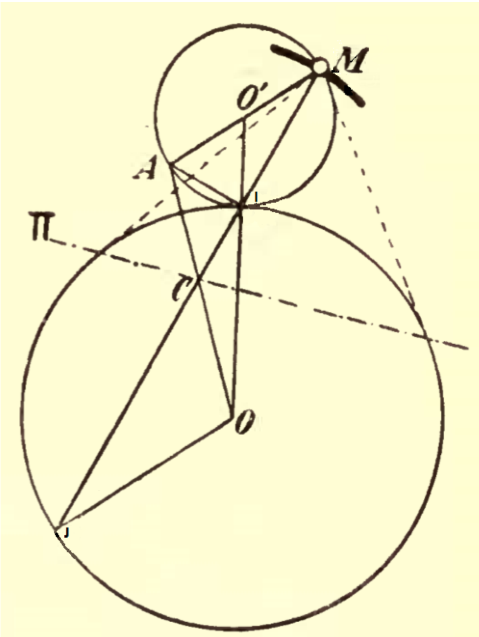

Figure 3: Property of radius of curvature for cycloidals : C is on the polar of M w.r.t the circle O.

With the previous equations we can compute the parameter $\tan V$:

$$\tan^2 V = \left(\frac{\rho d\theta}{d\rho}\right)^2 = \frac{R^2[1 - \frac{u^2+1}{u^2+m^2}]}{\frac{m^2R^2(u^2+1)}{u^2+m^2} - R^2} = \frac{1}{u^2}$$

$$\boxed{\tan^2 V = \tfrac{1}{u^2} \longrightarrow \tan V = \pm\tfrac{1}{u}} \qquad (15)$$

12 Angle V in cycloidal central polar coodinates.

Using the formula for angle V :

$$\tan V = \frac{\rho d\theta}{d\rho}$$

with central parametric equations of cycloidals :

$$\boxed{\left(\tfrac{\rho}{R}\right)^2 = \tfrac{1+u^2}{m^2+u^2} \qquad m = \tfrac{R}{R+2r}}$$

$$\boxed{\theta = \frac{1}{m}\arctan\frac{u}{m} - \arctan u}$$

We have :

$$2\rho d\rho = R^2\frac{2u(m^2-1)}{(m^2+u^2)^2}du$$

$$\frac{d\rho}{du} = \frac{R(m^2-1)u}{\sqrt{\frac{1+u^2}{m^2+u^2}}(m^2+u^2)^2}$$

$$\frac{d\theta}{du} = \left[\frac{1}{m^2+u^2} - \frac{1}{1+u^2}\right]$$

$$\tan V = \frac{R\sqrt{\frac{1+u^2}{m^2+u^2}}\left[\frac{1}{m^2+u^2} - \frac{1}{1+u^2}\right]}{\frac{R^2.u(m^2-1)}{R\sqrt{\frac{1+u^2}{m^2+u^2}}.(m^2+u^2)^2}} = -\frac{1}{u}$$

And finally :

$$\boxed{\tan V = -\tfrac{1}{u}}$$

The parameter u is equal to $\tan(V - \pi/2)$.

13 Generalization of cycloidals by Mc Laurin and pedal transformations and a tangent parameter k

We see that if we set $u = \tan\alpha$ we have $V = \pi/2 - \alpha$. The parameter $\pm 1/\tan\alpha$ is just $\tan V$ which is the parameterwe have used in Part III to define a generalization of sinusoidal spirals.

We call positive pedal of a curve (C, O) the curve generated by the projection H of O on the tangent at the end of ρ. This transformation preserves the angle V.
We can define the successive pedals indexed by a parameter p in N.The equation of the pedal is :

$$\rho_p = \rho \sin V \text{ and } \theta_p = \theta + (\pi/2 - V)$$

$$\rho_p = \rho \cos \alpha \text{ and } \theta_p = \theta + \alpha$$

For the pedal p becomes p+1 and θ becomes $\theta + \alpha$. The negative pedal (or anti-pedal) is the enveloppe of the perpendicular to the end of ρ. For the anti-pedal p becomes p-1. Mc Laurin transformation is just $Z \to z^n = \rho^n e^{in\theta}$

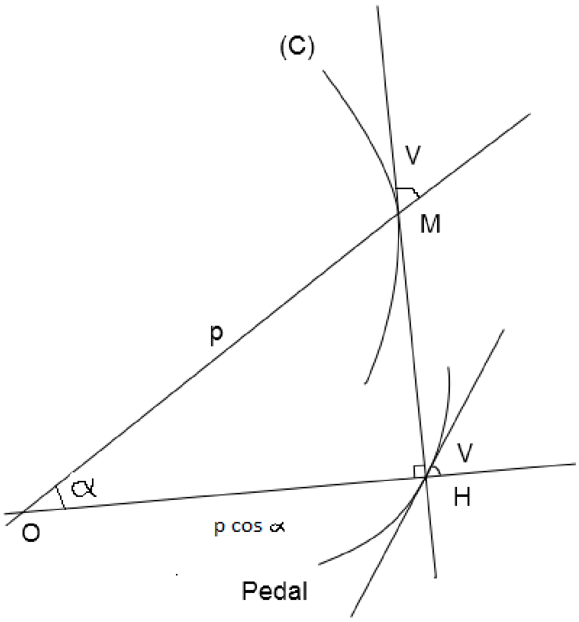

Figure 4: Pedal anti-pedal

so $\rho \to \rho^n$ and $\theta \to n\theta$. We assume here that n and p are integers $\in Z$. So if we modify an initial cycloidal by Mc Laurin (n) and successive pedal (p) or antipedal (-p) transformations we obtain new families exactly in the same way (m defined above precise the initial cycloidal and k is the tangent parameter).

$$\boxed{(\frac{\rho}{R})^2 = \left[\frac{(\cos \alpha)^{2(p-1)}}{(\tan \alpha)^2 + m^2}\right]^n} \quad (12)$$

$$\boxed{\theta = \frac{n}{m} \arctan \frac{\tan \alpha}{m} + n(p-1)\alpha} \quad (13)$$

With this equation and help of a mathematic software we verify that :

$$\tan V = -\frac{1}{\tan\alpha} = -\frac{1}{u}$$

13.1 Generalization of cycloidals. Curves $C_k(m, n, p)$.

Just as in Part III for Sinusoidal spirals and Ribaucour curves the cycloidals can be generalized by Mac Laurin transformation equivalent to $Z = z^n$ or $\rho \to \rho^n$ and $\theta \to n\theta$ and pedal transformation from O applying above formulas.
We impose that V is a linear multiple of α the parameter so $\tan V$ is equal to $\tan k\alpha$ (k) or its inverse $1/\tan k\alpha$ (k*):

$$\tan V = \frac{\rho d\theta}{d\rho} = \tan k\alpha \qquad \text{(k) or} \qquad = 1/\tan k\alpha \qquad (k*)$$

And we search for solutions (ρ is the unknown function and *alpha* the variable) of the differential equation :

$$\frac{d\rho}{\rho} = \frac{d\theta}{\tan k\alpha} \qquad \text{(k) or} \qquad \frac{d\rho}{\rho} = \tan k\alpha . d\theta \qquad (k*)$$

In these formulas θ is always equal to : $\frac{n}{m}\arctan\frac{\tan\alpha}{m} + n(p-1)\alpha$

We keep the formulas depending of n, p and m and define the generalized family of curves $C_k(m, n, p)$ where m=R/(R+2r), p=pedal index, n=McLaurin index and k the angle index.

The solutions for integers values of m, n, p and k are a generalization of cycloidals. For all these curves the angle V is a multiple of the parameter α so it is sometimes possible to find the arc length by elementary functions.

And for each value of integer k, with help of a mathematical software we find :

We set $u = \tan\alpha$ in the following.

$$\theta = \frac{n}{m}\arctan(\frac{\tan\alpha}{m}) + n(p-1)\alpha$$

k=1 : $\tan V = \tan\alpha$

$$(\frac{\rho}{R})^2 = [(\frac{u^2}{m^2+u^2})^{\frac{1}{m^2}}.(\frac{u^2}{1+u^2})^{p-1}]^n$$

k*=1 : $\tan V = 1/\tan\alpha$

$$(\frac{\rho}{R})^2 = [(1+u^2)^{p-1}.(m^2+u^2)]^n$$

These are tranformed of cycloidals combined with inversion centered at O :

$$(\frac{\rho}{R})^2 = \Big[\frac{(1+u^2)^{1-p}}{m^2+u^2}\Big]^n = \Big[\frac{(\cos\alpha)^{2(p-1)}}{m^2+\tan^2\alpha}\Big]^n$$

k=2 : $\tan V = \tan 2\alpha$

$$(\frac{\rho}{R})^2 = \Big[(\frac{2u}{1+u^2})^p.(\frac{u^2}{1+u^2})^{\frac{1-m^2}{2m^2}}.(\frac{1+u^2}{m^2+u^2})^{\frac{1+m^2}{2m^2}}\Big]^n$$

k*=2 : $\tan V = 1/\tan 2\alpha$

$$(\frac{\rho}{R})^2 = \Big[\Big(\frac{1-u^2}{1+u^2}\Big)^{\frac{m^2-1}{m^2+1}-p}.(\frac{m^2+u^2}{1+u^2})^{\frac{2}{1+m^2}}\Big]^n$$

The following values of k lead to heavy formulas. Cycloidals, Sinusoidal spirals and Ribaucour curves are in a coherent family with a good $\tan V$. Conics are different. Jacobi and Abel knew that.

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This article is the $XVII^{th}$ on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Two papers in french :

1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr