ASTROID - NEPHROID
DELTOID - CARDIOID
ORTHOCYCLOIDALS
- Part XVIII -

C. Masurel
08/01/2017

Abstract

Astroid and Nephroid are close curves with many relations. They are special examples of hypercycles proposed by Laguerre to generalize polarity in the plane for curves of class four by analogy with polarity for conics of class two. We present a transformation that links couples of orthocycloidals and curves transformed by central inversion and apply it to astroid/nephroid and deltoid/cardioid. We also give a proof of an old central polar equation of the Nephroid.

1 Epicycloids, hypocycloids and orthocycloidals

We have seen in part XIV that couples of ortho cycloidals can move inside a couple of epi- and hypo-cycloids in such a way that all cusps stay on epi- and hypo-cycloids and the curves pass through their common cusps and keep orthogonal crossing.

F. Morley in [6] indicates that ”the only points at infinity of cycloidals of degree 2p are the cyclic points I and J counted each p times. In the epicycloid the singular tangents at I and J are directed to the origin, where all the foci are collected. The center being the mean of the foci, the origin is also the center of the curve. In the epicycloid the line IJ is the singular tangent at both I and J, and there are no finite foci.”

This is an incitement to study cycloidals in polar coordinates with O at the center of the fixed circle of the cycloidal.

So we examine some special cases of orthocycloidals using central polar equation of cycloidals of F. Gomez- Teixeira.

A simple definition of cycloidals by the general transformation given in complex plane $z = x + iy$ by the formula :

$$Z = z^n + k.z = X + iY$$
Cycloidals are transformed of a circle: \( z = e^{it} \) (t is an angle parameter) so 
\[ Z = ke^{it} + e^{ikt} = X + iY. \]
In usual parametric coordinates:

\[
X_E(t) = k \cos t + \cos kt \\
Y_E(t) = k \sin t + \sin kt \\
X_H(t) = k \cos t + \cos kt \\
Y_H(t) = k \sin t - \sin kt
\]

Some signs have been changed for (H) so that the two equal rolling circles coincide and the generation point is on the Ox line at intersection with external circle (at a cusp for Hypocycloid and at a summit for the epicycloid).

Note that these roll in opposite rotation. If \( k > 0 \) we get epicycloids and for \( k < 0 \) we have hypocycloids. A couple of orthocycloidals corresponds to opposite values of \( k \). \( k = \pm 2 \) : Cardioid/Deltoid, \( k = \pm 3 \) : Nephroid/Astroid, etc.

Figure 1: Examples of orthocycloidals in a circular annulus: Cardioid/Deltoid, Nephroid/Astroid, \( k = \pm 4 \) and \( k = \pm 2/3 \).

2 Astroid

This curve, known since Roemer in 1674, we will just recall some facts and equations. It belongs to the family of Lame’s curves:

\[
x = a \cos t^3 \\
y = a \sin t^3 \\
x^{2/3} + y^{2/3} = a^{2/3}
\]

The astroid is a 4-hypocycloid for \( R = 4x \):

\[
x = \frac{a}{4}(3 \cos t + \cos 3t) \\
y = \frac{a}{4}(3 \sin t - \sin 3t)
\]

Tangential equation: \((u^2 + v^2).w^2 - au^2v^2 = 0\)

or: \(\frac{1}{u^2} + \frac{1}{v^2} = \frac{a}{w^2}\)

Intrinsic equation of cesaro \( f(R,s)=0 \):

\[ R^2 - 4s^2 = \frac{9}{4}a^2 \]

The pedal equation pole at the center:

\[ \rho^2 = a^2 - 3p^2 \]
3 The Nephroid

This curve was known since Huygens about 1670, we just recall some facts and equations. Parametric equations:

\[ x = 4a \sin t^3 \quad y = 2a(1 + 2 \sin t^2) \cos t \]
The nephroid is a 2-epicycloid for $R = 2r$:

$$
x = a(3 \sin t + \sin 3t) \quad y = a(3 \cos t + \cos 3t)
$$

Tangential equation: $(u^2 + v^2) f(u, v)^2 - a \Phi(u, v)^2 = 0$

Intrinsic equation of cesaro $f(R, s)$: $4R^2 + s^2 = 36a^2$

The pedal equation pole at the center:

$$4\rho^2 - 3p^2 = 16a^2$$

Figure 4: Astroid and Nephroid as a couple of ortho-cycloids.

4 Central parametric equations of cycloids

In Tome 3 of [1] Gomez Teixeira gives parametric equations in polar coordinates of cycloids when the pole is at the center of the director circle. $\rho$ is the vector radius and $\theta$ the polar angle and a parameter $u$ given by expression:

$$u = \sqrt{\frac{m^2 \rho^2 - R^2}{R^2 - \rho^2}}$$

$R$ and $r$ are the radii of fixed director and rolling circles ($R, r > 0$): $+r$ for epicycloids and $-r$ for hypocycloids.

$$\left(\frac{\rho}{R}\right)^2 = \frac{u^2 + 1}{u^2 + m^2} \quad m = \frac{R}{R + 2r} \quad (1)$$
The value $m > 1$ for the hypocycloid since $r < 0$ (circle rolling inside director circle) and $m < 1$ for the epicycloid (circle rolling outside) in a same couple of orthocycloids. The circular annulus $[R_i, R_e]$ contains the two curves which stay orthogonal when rotated around common center and $m$ is equal to $\frac{R_e}{R_i}$. Cusps of the hypocycloid are on external circle and summits are in the internal. For the epicycloid cusps are on internal and summits on the external one. Polar angle is:

$$\theta = \frac{1}{m} \arctan \frac{u}{m} - \arctan u$$ \hspace{1cm} (2)

The equations of these associated cycloidals hypo-1 and epi-2 are:

(here we use the parameter $k = 1 + \frac{R}{r}$ so $m = \frac{k-1}{k+1}$, the angular position $t$ is the one of the line joining the center of the circular annulus to the center of the rolling circle).

$$\begin{align*}
X_1(t) &= k \cos t + \cos kt \\
Y_1(t) &= k \sin t - \sin kt
\end{align*}$$

$$\begin{align*}
X_2(t) &= k \cos t + \cos kt \\
Y_2(t) &= k \sin t + \sin kt
\end{align*}$$

For hypo-cycloidal 1 we have the following formulas:

$$\begin{align*}
X_1(t) &= k \cos t + \cos kt \\
Y_1(t) &= k \sin t - \sin kt \\
\rho_1^2 &= 1 + k^2 + 2k \cos(t + kt) \\
\tan V &= \frac{k - 1}{k + 1} \frac{1}{\tan \left[ \frac{(1+k)t}{2} \right]} \\
ds_1 &= k \sin \left( \frac{1 + k}{2} t \right) dt
\end{align*}$$

Equation so the inverse $(O, m.R^2)$ curve are:

$$\begin{align*}
X_{11} &= m \frac{k \cos t + \cos kt}{1 + k^2 + 2k \cos(t + kt)} \\
Y_{11} &= m \frac{k \sin t - \sin kt}{1 + k^2 + 2k \cos(t + kt)} \\
\rho^2 &= \frac{m^2}{1 + k^2 + 2k \cos(t + kt)} \\
\tan V &= \frac{1 - k}{1 + k} \frac{1}{\tan \left[ \frac{(1+k)t}{2} \right]} \\
ds &= \frac{m.k \sin \left( \frac{1+k}{2} t \right)}{1 + k^2 + 2k \cos(t + kt)} \cdot dt
\end{align*}$$
And for epi-cycloidal 2 the formulas are:

\[ X_2(t) = k \cos t + \cos kt \]
\[ Y_2(t) = k \sin t + \sin kt \]
\[ \rho^2 = 1 + k^2 + 2k \cos(t - kt) \quad \tan V = \frac{k + 1}{k - 1} \frac{1}{\tan \left( \frac{1-k}{2} t \right)} \]
\[ ds_2 = k \cos \left( \frac{1-k}{2} t \right).dt \]

Equation so the inverse \((O, mR^2)\) curve are:

\[ X_{i2}(t) = m \frac{k \cos t + \cos kt}{1 + k^2 + 2k \cos(t - kt)} \]
\[ Y_{i2}(t) = m \frac{k \sin t + \sin kt}{1 + k^2 + 2k \cos(t - kt)} \]
\[ \rho_{i2}^2 = \frac{m^2}{1 + k^2 + 2k \cos(t - kt)} \quad \tan V = \frac{1 + k}{1 - k} \frac{1}{\tan \left( \frac{1-k}{2} t \right)} \]
\[ ds_2 = \frac{m.k \cos \left( \frac{1-k}{2} t \right)}{1 + k^2 + 2k(1 + \cos(t - kt))} dt \]

The relations between \(k\) and \(m\) is \(km+(m-k)+1=0\):

\[ m = \frac{k - 1}{k + 1} \quad k = \frac{1 + m}{1 - m} \]

For deltoid \(k= -2\) or \(m=3\), cardioid \(k= +2\) or \(m'=1/3\), astroid \(k=-3\) or \(m=2\) and nephroid \(k=+3\) or \(m'=1/2\).

In general for two orthocycloidals: hypocycloid 1 for \(k = k\) or \(m = m\) and associated epicycloid 2 for \(k' = -k\) or \(m' = 1/m\).

5 Central polar equation of the Nephroid

In old books and on the web we can find the following polar central equation of the nephroid:

\[ \left( \frac{\rho}{2r} \right)^{2/3} = \left( \cos \frac{\theta}{2} \right)^{2/3} + \left( \sin \frac{\theta}{2} \right)^{2/3} \]  \(3\)

This formula has for a long time been a mystery to me since I found it in [2] in 1977. It goes back at minima to 1952 in the book of R.C Yates probably much earlier. If someone knows where it has been first mentioned or proved I would be interested. We now present a proof of this polar central equation in two steps.
5.1 Central polar parametric equations of the Nephroid and of the Astroid (step 1)

We have seen in part XV that the astroid and the nephroid form a couple of orthocycloidals that can rotate around the common center inside a circular annulus and stay orthogonal. And it is possible to move the two curves between the 3-epicycloid and the deltoid so that the two curves pass through the three common cusps of these last two curves.

For astroid $m=2$ we have the following equations:

\[
\left(\frac{\rho_A}{2R}\right)^2 = \frac{v^2+1}{v^2+4} \quad \tan V = -1/v \quad m = 2
\]

\[
\theta_A = \frac{1}{2} \arctan \frac{v}{2} - \arctan v
\]

For nephroid $m=1/2$ so we have the following equations:

\[
\left(\frac{\rho_N}{R}\right)^2 = \frac{4(u^2+1)}{4u^2+1} \quad \tan V = -1/u \quad m' = 1/2 = 1/m
\]

\[
\theta_N = 2 \arctan 2u - \arctan u
\]

We see that the range of the vector radius is in $[2R, 4R]$ for the two expressions of $\rho$ so the two curves are inside the circular annulus and we know that they are constantly orthogonal for any rotation around the common center $O$ of the curves. If in the parametric equations of the astroid we use the change of variable $v=2u$ then:

\[
\left(\frac{\rho_A}{2R}\right)^2 = \frac{v^2+1}{v^2+4} = \frac{4.\frac{1}{2}u^2+1}{4.\frac{1}{2}u^2+1} = \left[\frac{u^2+1}{\left(u^2+\frac{1}{4}\right)}\right]^{-1} = \left(\frac{R}{\rho_N}\right)^2
\]

The polar angle for the astroid is : $\theta_A = \frac{1}{2} \arctan \frac{v}{2} - \arctan v$ (9)

With $v=2u$ we find : $\theta_A = \frac{1}{2} \arctan u - \arctan 2u$ (10)

for the nephroid the polar angle is : $\theta_N = 2 \arctan 2u - \arctan u$ (11)

So we see that:

\[
\theta_N = -2.\theta_A
\]

5.2 Central inverse of the astroid (step 2)

We have proved on one hand that $\rho_A.\rho_N = 2.R^2$ for corresponding points and on the other hand that there is an angular dilation which
transforms the inverse of the nephroid (so with same $\theta_N$) in the astroid with $\theta_A = -\frac{1}{2} \theta_N$.

In a similar way the central inverse of the astroid can be transformed into the Nephroid by angular dilation $\theta_N = -2 \theta_A$.

The four curves are inside the circular annulus $(O, R \rightarrow 2R)$. The reciprocal polar of a curve is the inverse of the pedal and is also the antipedal of the inverse. There is a circular graph: initial curve $\rightarrow$ pedal $\rightarrow$ inverse $\rightarrow$ pedal $\rightarrow$ initial curve. A classical result - see [4] - is that for a Lame curve: $(C^n) x^n + y^n = a^n$ the reciprocal polar (for a circle centered at $O$) is another Lame curve:

$$x^{\frac{n}{n-1}} + y^{\frac{n}{n-1}} = a^{\frac{n}{n-1}} \quad (13)$$

And the pedal ($O$ at the center) of a Lame curves $(C^n)$ is:

$$(\rho/a)^{\frac{n}{n-1}} = (\cos \theta)^{\frac{n}{n-1}} + (\sin \theta)^{\frac{n}{n-1}} \quad (14)$$

For the astroid $n = \frac{2}{3}$ the reciprocal polar with $(\frac{n}{n-1} = -2)$ is:

$$x^{-2} + y^{-2} = a^{-2} \quad \text{the Crosscurve} \quad (15)$$

Using formula (14) with $a = 2R$ the pedal of $(C^{-2})$ is:

$$(\rho/2R)^{\frac{3}{1}} = (\cos \theta_A)^{\frac{3}{1}} + (\sin \theta_A)^{\frac{3}{1}} \quad (16)$$

Which is the inverse of the astroid by the property of the reciprocal polar mentioned above. We use now the angular dilation $\theta_N = -2 \theta_A$ (step 1) to transform the central inverse of the astroid in the nephroid. This gives finally the central polar equation of the nephroid (3):

$$\left(\frac{\rho}{2R}\right)^{2/3} = \left(\cos \frac{\theta_N}{2}\right)^{2/3} + \left(\sin \frac{\theta_N}{2}\right)^{2/3}$$

5.3 Parametric equations of Astroid, nephroid and their inverses

Equations of astroid usual parameter are:

$$x_a = 3 \cos t + \cos 3t = 4 \cos^3 t$$
$$y_a = 3 \sin t - \sin 3t = 4 \sin^3 t$$
$$\rho^2 = 10 + 6 \cos 4t = 16(\cos^6 t + \sin^6 t) \quad ds = 6 \sin 2t.dt$$

Equations of the central inverse $(O, 2.R^2)$ curve are:

$$X_{ia} = \frac{16 \cos^3 t}{5 + 3 \cos 4t}$$
\[ Y_{\text{na}} = \frac{16 \sin^3 t}{5 + 3 \cos 4t} \]
\[ \rho^2 = \frac{32}{5 + 3 \cos 4t} \quad ds = \frac{24 \sin 2t}{5 + 3 \cos 4t} \, dt \]

Equations of nephroid t usual parameter are :
\[
x_c = 3 \cos t + \cos 3t = 2(1 + 2 \cos^2 t) \sin t
\]
\[
y_c = 3 \sin t + \sin 3t = 4 \cos^3 t
\]
\[ \rho^2 = 10 + 6 \cos 2t \quad ds = 6 \sin t \, dt \]

Equations of the inverse \((O, 8R^2)\) curve are :
\[
X_{\text{id}} = \frac{4 \sin t(2 + \cos 2t)}{5 + 3 \cos 2t}
\]
\[
Y_{\text{id}} = \frac{8 \cos^3 t}{5 + 3 \cos 2t}
\]
\[ \rho^2 = \frac{8}{5 + 3 \cos 2t} \quad ds = \frac{12 \cos t}{5 + 3 \cos 2t} \, dt \]

The central inverse of the astroid and the central inverse of nephroid are also ortho-curves for any rotation around the pole O.

6 Generalization of the angular dilation property to cycloidals

6.1 Angular homothetic of the central inverse of couple of orthocycloidals

The above computation for the couple astroid/nephroid can be generalized to any couple of orthocycloidals. We see that the range of the vector radius is in \([1, m]\) or \([R, m.R]\) for the two expressions of \(\rho\) so the two orthocycloidals are inside the circular annulus and we know that they are constantly orthogonal for any rotation around the common center O of the curves. The inversions we use in the paper exchange the two border-circles of the annulus. We use parameter \(m = \frac{R}{R+2r} > 1\) with \(r < 0\) for the hypocycloidal and \(m' = \frac{1}{m}\) for the ortho-associated epicycloidal. In the parametric equations of the cycloidals we use the change of variable \(v=m.u\) then :
\[
\left( \frac{\rho_1}{mR} \right)^2 = \frac{u^2 + 1}{v^2 + m^2} = \frac{m^2.u^2 + 1}{m^2.u^2 + m^2} = \left[ \frac{u^2 + 1}{u^2 + \frac{1}{m^2}} \right]^{-1} = \left( \frac{R}{\rho_2} \right)^2 \quad (16)
\]
The polar angles are for the hypocycloid 1:

\[ \theta_1 = \frac{1}{m} \arctan \frac{v}{m} - \arctan v \quad (17) \]

we use \( v = m \cdot u \) to transform the first hypocycloid angle and find:

\[ \theta_1 = \frac{1}{m} \arctan u - \arctan (m \cdot u) \quad (18) \]

for the epicycloid 2 : \( m' = \frac{1}{m} \), the polar angle is:

\[ \theta_2 = m \arctan m \cdot u - \arctan u \quad (19) \]

So we see that:

\[ \theta_2 = -m \theta_1 \quad (20) \]

The similarity with couple astroid/nephroid permits only to define the transformation that applies the two orthocycloidal one on the other. If we begin with the hypocycloid 1 : find the central inverse \((O, m \cdot R^2)\) then apply the angular dilation (20) we get the orthocycloidal : epicycloid 2. Conversely if we begin with epicycloid 2, inversion combined with angular dilation give hypocycloid 1. The central inverse of hypocycloid 1 and the central inverse of epicycloid 2 are also orthocurves for any rotation around the pole O.

6.2 Orthogonal trajectories of cycloidal in central polar coordinates

We examine orthogonal trajectories for rotations of cycloidal around the pole O center of the director circle. Using central parametric polar coordinates of Gomez Teixeira.

\[ \left( \frac{\rho}{R} \right)^2 = \frac{u^2 + 1}{u^2 + m^2} \quad m = \frac{R}{R + 2r} \]

\[ \theta = \frac{1}{m} \arctan \frac{u}{m} - \arctan u \]

We know that for cycloidal \( \tan V = -\frac{1}{u} \) so at point of intersection \((\rho = \rho_\perp)\) of the orthogonal curve \( \tan V_\perp = u \) so \( \tan V, \tan V_\perp = -1 \) for orthogonality. We have (setting \( R=1)\):

\[ \tan V_\perp = \frac{\rho^2 d\theta_\perp}{\rho \cdot dp} \rightarrow d\theta_\perp = \frac{\rho \cdot dp}{\rho^2} = \frac{(m^2 - 1)u^2 \cdot du}{(u^2 + m^2)(u^2 + 1)} \]
We find :

\[ \theta_\perp = m \cdot \arctan \frac{u}{m} - \arctan u \quad \rho_\perp^2 = \frac{u^2 + 1}{u^2 + m^2} \]

The expression of the angle \( \theta_\perp \) can be transformed using the same changes of variable as above \( v = m.u \) :

\[ \theta_\perp = -m \left[ \frac{1}{m} \arctan \frac{v}{m} - \arctan v \right] = -m \theta \]

\[ \rho_\perp^2 = \frac{v^2 + 1}{v^2 + m^2} = \frac{m^2.u^2 + 1}{m^2.u^2 + m^2} = \frac{u^2 + 1}{u^2 + 1} = \frac{u^2 + m^2}{u^2 + 1} = \frac{1}{\rho_\perp^2} \quad m' = 1/m \]

It confirms the result found in section above on couple orthocycloidals and couple of inverse w.r.t. the center O.

7 Another example of orthocycloidals:

**Deltoid and Cardioid**

The deltoid and the cardioid are a couple of orthocycloidals for \( m=3 \) and \( m'=1/3 \). By the same argument as before we have a square of curves using the two transformations: central inversion in \( \rho \) and angular dilation around O with parameter: \( m=3 \). So we set \( v=3.u \).

\[ \left( \frac{\rho_1}{3R} \right)^2 = \frac{v^2 + 1}{v^2 + 3^2} = \frac{3^2.u^2 + 1}{3^2.u^2 + 3^2} = \left[ \frac{u^2 + 1}{u^2 + \frac{1}{3^2}} \right]^{-1} = \left( \frac{R}{\rho_2} \right)^2 \]

The polar angles are for the deltoid: \( \theta_1 = \frac{1}{3} \arctan \frac{v}{3} - \arctan v \quad (21) \)

we use \( v=3.u \) to transform the deltoid angle and find :

\[ \theta_1 = \frac{1}{3} \arctan u - \arctan 3.u \quad (22) \]

for the cardioid the polar angle is :

\[ \theta_2 = 3 \arctan 3.u - \arctan u \quad (23) \]

So we see that :

\[ \theta_2 = -3 \theta_1 \quad (24) \]
7.1 Parametric equations of deltoid, cardioid and their inverses

Equations of the deltoid with t usual parameter are:

\[ x_d = 2 \cos t + \cos 2t = 2(1 + \cos t) \cos t - 1 \]
\[ y_d = 2 \sin t - \sin 2t = 2(1 - \cos t) \sin t \]
\[ \rho^2 = 5 + 4 \cos 3t \quad ds = 2 \sin(3t/2) dt \]

Equations of the inverse \((O,3.R^2)\) curve are:

\[ X_{id} = \frac{3(2 \cos t + \cos 2t)}{5 + 4 \cos 3t} \]
\[ Y_{id} = \frac{3(2 \sin t - \sin 2t)}{5 + 4 \cos 3t} \]
\[ \rho^2 = \frac{9}{5 + 4 \cos 3t} \quad ds = \frac{12 \sin(3t/2)}{5 + 4 \cos 3t} dt \]

Equations of cardioid with t usual parameter are:

\[ x_c = 2 \cos t + \cos 2t = 2(1 + \cos t) \cos t - 1 \]
\[ y_c = 2 \sin t + \sin 2t = 2(1 + \cos t) \sin t \]
\[ \rho^2 = 5 + 4 \cos t \quad ds = 2 \cos(t/2) dt \]
Equations of the inverse \((O, 3R^2)\) curve are:

\[
\begin{align*}
X_{id} &= \frac{3(2 \cos t + \cos 2t)}{5 + 4 \cos t} \\
Y_{id} &= \frac{3(2 \sin t + \sin 2t)}{5 + 4 \cos t} \\
\rho^2 &= \frac{9}{5 + 4 \cos t} \\
ds &= \frac{12 \cos t/2}{5 + 4 \cos t} \cdot dt
\end{align*}
\]

The central inverse of the deltoid and the central inverse of cardioid are also orthocurves for any rotation around the pole \(O\).

Figure 6: The Deltoid and central inverse \((m=3)\)

8 Orthocycloidals as antipedals of rhodoneas: \(\rho = \cos m.\theta\) and \(\rho = \cos \left(\frac{\theta}{m}\right)\)

Antipedal of rhodoneas are cycloidals: hypocycloids when \(m > 1\) and epicycloids when \(m < 1\).

In [5] J. Lemaire presents a tangential generation of cycloidals in a simple manner. Given a fixed circle \((O, R)\) and on it an origin of angles \(S\). A first point \(P\) is at angle position \(t\) (the parameter of our curves) and second point \(Q\) is at angle position \(h.t\) (\(h\) real number). \(h > 0\) for epicycloids and \(h < 0\) for hypocycloid (J. Lemaire uses the opposit values for \(h\)). The envelope of line \(PQ\) when \(t\) varies is a cycloidal. With this geometric tangential definition we easily find the equation of the pedal since:

\[
\rho = R \cos \frac{h - 1}{2} t \\
\theta = \frac{h + 1}{2} t
\]
And antipedal of these rodhoneas are cycloidals (see Part XI).

$h < 0$ Hypocycloids, $h > 0$ Epicycloids, $h = 1$ a circle and $h = -1$ the 2-hypocycloid or a segment (degenerated). $h = 2$: Cardioid, $h = -2$: Deltoid, $h = 3$: Nephroid, $h = -3$: Astroid, etc. If we change $h$ in $-h$ then the rodhena is $\rho = R \cos \frac{h-1}{h+1} t = R \cos \frac{1}{m} t$ and becomes $\rho = R \cos \frac{h}{h+1} t = R \cos \frac{1}{m} t$. For two orthocycloidals: $m = m$ and $m' = \frac{1}{m}$ or $m = \frac{h+1}{h-1}$ and $m' = \frac{h-1}{h+1}$. So parameter $h$ is the same as the $k$ used in the formulas for parametric cycloidals in section 3 above.
Figure 9: Orthocurves for rotation (transformed of astroid and deltoid by central inversion) m=2

References:
- [1] F. Gómez-Teixeira, Traité des courbes planes remarquables (Tomes 2 pp 273-274 3 pp 175, 180)

Part I: Gregory’s transformation.
Part II: Gregory’s transformation Euler/Serret curves with same arc length as the circle.
Part III: A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen’s cubic.
Part V: Closed wheels and periodic grounds
Part VI: Catalan’s curve.
Part VII: Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves.
Part VIII: Translations, rotations, orthogonal trajectories, differential equations, Gregory’s transformation.
Part IX: Curves of Duporcq - Sturmian spirals.
Part X: Intrinsically defined plane curves, periodicity and Gregory’s transformation.
Part XI: Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d’Ocagne axial coordinates.
Part XII: Caustics by reflection, curves of direction, rational arc length.
Part XIII: Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV: Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV: Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI: Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII: Cesaro’s curves - A generalization of cycloidals.
Part XVIII: Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Two papers in french:
1- Quand la roue ne tourne plus rond - Bulletin de l’IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l’APMEP (no 364 juin 1988).
Gregory’s transformation on the Web: http://christophe.masurel.free.fr