ASTROID - NEPHROID DELTOID - CARDIOID ORTHOCYCLOIDALS - Part XVIII -

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Abstract

Astroid and Nephroid are close curves with many relations. They are special examples of hypercycles proposed by Laguerre to generalize polarity in the plane for curves of class four by analogy with polarity for conics of class two. We present a transformation that links couples of orthocycloidals and curves transformed by central inversion and apply it to astroid/nephroid and deltoid/cardioid. We also give a proof of an old central polar equation of the Nephroid.

1 Epicycloids, hypocycloids and orthocycloidals

We have seen in part XIV that couples of ortho cycloidals can move inside a couple of epi- and hypo-cycloids in such a way that all cusps stay on epi- and hypo-cycloids and the curves pass through their common cusps and keep orthogonal crossing.
F. Morley in [6] indicates that "the only points at infinity of cycloidals of degree 2p are the cyclic points I and J counted each p times. In the epicycloid the singular tangents at I and J are directed to the origin, where all the foci are collected. The center being the mean of the foci, the origin is also the center of the curve. Inthe epicycloid the line IJ is the singular tangent at both I and J, and there are no finite foci."
This is an incitement to study cycloidals in polar coordinates with O at the center of the fixed circle of the cycloidal.
So we examine some special cases of orthocycloidals using central polar equation of cycloidals of F. Gomez- Teixeira.
A simple definition of cycloidals by the general transformation given in complex plane $z = x + iy$ by the formula :

$$Z = z^n + k.z = X + iY$$

Cycloidals are transformed of a circle : $z = e^{i.t}$ (t is an angle parmeter) so $Z = ke^{it} + e^{ikt} = X + iY$. In usual parametric coordinates :

$$X_E(t) = k\cos t + \cos kt \qquad Y_E(t) = k\sin t + \sin kt$$

$$X_H(t) = k\cos t + \cos kt \qquad Y_H(t) = k\sin t - \sin kt$$

Some signs have been changed for (H) so that the two equal rolling circles coincide and the generation point is on the Ox line at intersection with external circle (at a cusp for Hypocycloid and at a summit for the epicycloid). Note that these roll in opposite rotation. If $k > 0$ we get epicloids and for $k < 0$ we have hypocycloids. A couple of orthocycloidals corresponds to opposite values of k. $k = \pm 2$: Cardioid/Deltoid, $k = \pm 3$: Nephtoid/Astroid, etc.

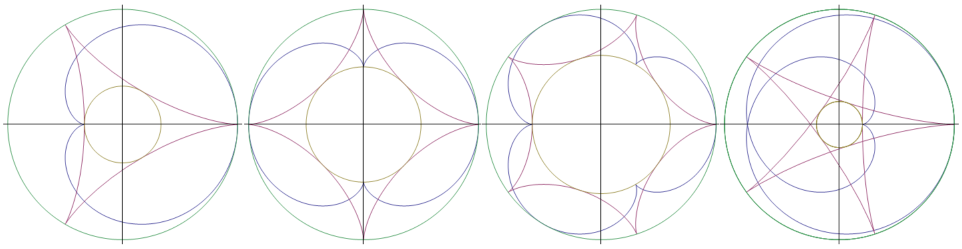

Figure 1: Examples of orthocycloidals in a cicular annulus : Cardioid/ Deltoid, Nephroid/Astroid, $k = \pm 4$ and $k = \pm 2/3$.

2 Astroid

This curve, known since Roemer in 1674, we will just recall some facts and equations. It belongs to the family of Lame's curves :

$$x = a\cos t^3 \qquad y = a\sin t^3$$

$$x^{2/3} + y^{2/3} = a^{2/3}$$

The astroid is a 4-hypocycloid for $R = 4.r$:

$$x = \frac{a}{4}(3\cos t + \cos 3t) \qquad y = \frac{a}{4}(3\sin t - \sin 3t)$$

Tangential equation : $\qquad (u^2 + v^2).w^2 - au^2v^2 = 0$

or : $\qquad \frac{1}{u^2} + \frac{1}{v^2} = \frac{a}{w^2}$

Intrinsic equation of cesaro f(R,s)=0 : $\qquad R^2 - 4s^2 = \frac{9}{4}a^2$

The pedal equation pole at the center :

$$\rho^2 = a^2 - 3p^2$$

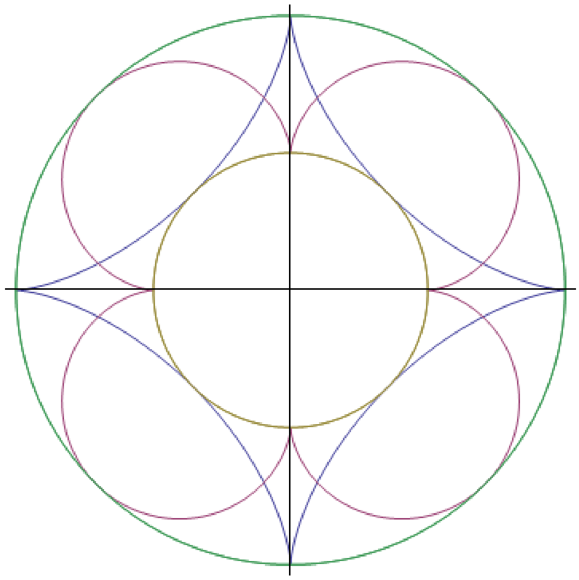

Figure 2: In a circular annulus : Astroid and its central inverse

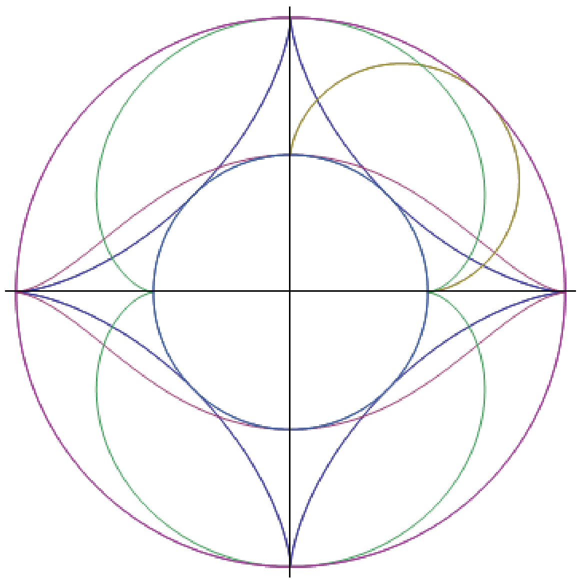

Figure 3: In a circular annulus : Nephroid and its central inverse (+ Astroid and 1 arch of central inverse)

3 The Nephroid

This curve was known since Huygens about 1670, we just recall some facts and equations. Parametric equations :

$$x = 4.a \sin t^3 \qquad y = 2.a(1 + 2\sin t^2)\cos t$$

The nephroid is a 2-epicycloid for $R = 2.r$:

$$x = a(3\sin t + \sin 3t) \qquad y = a(3\cos t + \cos 3t)$$

Tangential equation : $\quad (u^2 + v^2).f(u,v)^2 - a\Phi(u,v)^2 = 0$

Intrinsic equation of cesaro f(R,s) : $\quad 4.R^2 + s^2 = 36.a^2$

The pedal equation pole at the center :

$$4\rho^2 - 3p^2 = 16.a^2$$

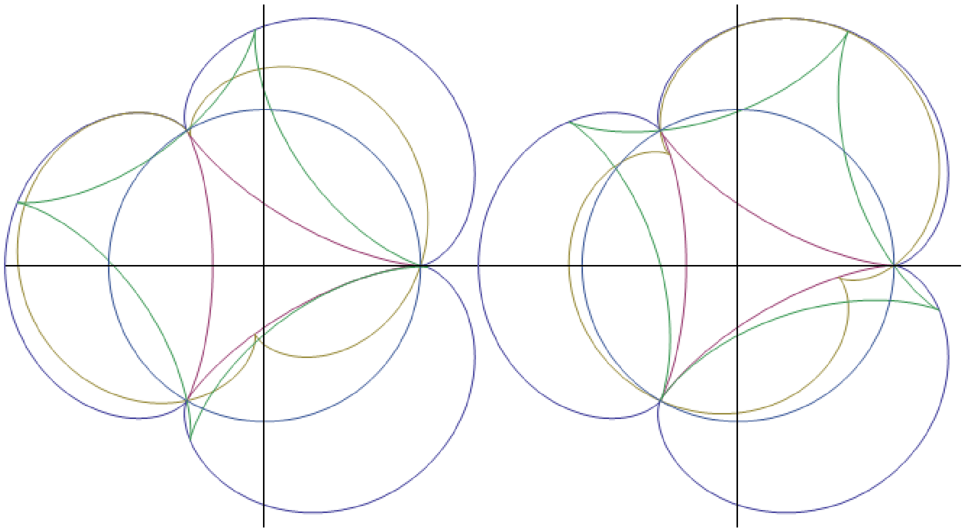

Figure 4: Astroid and Nephroid as a couple of ortho-cycloidals.

4 Central parametric equations of cycloidals

In Tome 3 of [1] Gomez Teixeira gives parametric equations in polar coordinates of cycloidals when the pole is at the center of the director circle. ρ is the vector radius and θ the polar angle and a parameter u given by expression :

$$\boxed{u = \sqrt{\frac{m^2\rho^2 - R^2}{R^2 - \rho^2}}}$$

R and r are the radii of fixed director and rolling circles $(R, r > 0)$: +r for epicycloids and -r for hypocycloids.

$$\boxed{\left(\frac{\rho}{R}\right)^2 = \frac{u^2+1}{u^2+m^2} \qquad m = \frac{R}{R+2r}} \qquad (1)$$

The value $m > 1$ for the hypocycloid since $r < 0$ (circle rolling inside director circle) and $m < 1$ for the epicycloid (circle rolling outside) in a same couple of orthocycloidals. The circular annulus $[R_i, R_e]$ contains the two curves which stay orthogonal when rotated around common center and m is equal to $\frac{R_e}{R_i}$. Cusps of the hypocycloid are on external circle and summits are in the internal. For the epicycloid cusps are on internal and summits on the external one. Polar angle is :

$$\boxed{\theta = \frac{1}{m} \arctan \frac{u}{m} - \arctan u} \qquad (2)$$

The equations of these associated cycloidals hypo-1 and epi-2 are : (here we use the parameter $k = 1 + \frac{R}{r}$ so $m = \frac{k-1}{k+1}$, the angular position t is the one of the line joining the center of the circular annulus to the center of the rolling circle).

$$\boxed{X_1(t) = k \cos t + \cos kt \qquad Y_1(t) = k \sin t - \sin kt}$$

$$\boxed{X_2(t) = k \cos t + \cos kt \qquad Y_2(t) = k \sin t + \sin kt}$$

For hypo-cycloidal 1 we have the following formulas :

$$X_1(t) = k \cos t + \cos kt$$
$$Y_1(t) = k \sin t - \sin kt$$

$$\rho_1^2 = 1 + k^2 + 2k \cos(t + kt) \qquad \tan V = \frac{k-1}{k+1} \frac{1}{\tan[\frac{(1+k)}{2} t]}$$

$$ds_1 = k \sin(\frac{1+k}{2} t) dt$$

Equation so the inverse $(O, m.R^2)$ curve are :

$$X_{i1} = m \frac{k \cos t + \cos kt}{1 + k^2 + 2k \cos(t + kt)}$$

$$Y_{i1} = m \frac{k \sin t - \sin kt}{1 + k^2 + 2k \cos(t + kt)}$$

$$\rho^2 = \frac{m^2}{1 + k^2 + 2k \cos(t + kt)} \qquad \tan V = \frac{1-k}{1+k} \frac{1}{\tan[\frac{(1+k)}{2} t]}$$

$$ds = \frac{m.k \sin(\frac{1+k}{2} t)}{1 + k^2 + 2k \cos(t + kt)} .dt$$

And for epi-cycloidal 2 the formulas are :

$$X_2(t) = k\cos t + \cos kt$$
$$Y_2(t) = k\sin t + \sin kt$$
$$\rho^2 = 1 + k^2 + 2k\cos(t - kt) \qquad \tan V = \frac{k+1}{k-1}\frac{1}{\tan[\frac{(1-k)}{2}t]}$$
$$ds_2 = k\cos(\frac{1-k}{2}t).dt$$

Equation so the inverse $(O, m.R^2)$ curve are :

$$X_{i2}(t) = m\frac{k\cos t + \cos kt}{1 + k^2 + 2k\cos(t - kt)}$$
$$Y_{i2}(t) = m\frac{k\sin t + \sin kt}{1 + k^2 + 2k\cos(t - kt)}$$
$$\rho_2^2 = \frac{m^2}{1 + k^2 + 2k\cos(t - kt)} \qquad \tan V = \frac{1+k}{1-k}\frac{1}{\tan[\frac{(1-k)}{2}t]}$$
$$ds_2 = \frac{m.k\cos(\frac{1-k}{2}t)}{1 + k^2 + 2k(1 + \cos(t - kt))}dt$$

The relations between k and m is km+(m-k)+1=0 :

$$m = \frac{k-1}{k+1} \qquad k = \frac{1+m}{1-m}$$

For deltoid k=-2 or m=3, cardioid k=+2 or m'=1/3, astroid k=-3 or m=2 and nephroid k=+3 or m'=1/2.
In general for two orthocycloidals : hypocycloid 1 for $k = k$ or $m = m$ and associated epicycloid 2 for $k' = -k$ or $m' = 1/m$.

5 Central polar equation of the Nephroid

In old books and on the web we can find the following polar central equation of the nephroid :

$$\boxed{\left(\frac{\rho}{2R}\right)^{2/3} = \left(\cos\frac{\theta}{2}\right)^{2/3} + \left(\sin\frac{\theta}{2}\right)^{2/3}} \qquad (3)$$

This formula has for a long time been a mystery to me since I found it in [2] in 1977. It goes back at minima to 1952 in the book of R.C Yates probably much earlier. If someone knows where it has been first mentioned or proved I would be interested. We now present a proof of this polar central equation in two steps.

5.1 Central polar parametric equations of the Nephroid and of the Astroid (step 1)

We have seen in part XV that the astroid and the nephroid form a couple of orthocycloidals that can rotate around the common center inside a circular annulus and stay orthogonal. And it is possible to move the two curves between the 3-epicycloid and the deltoid so that the two curves pass through the three common cusps of these last two curves.
For astroid m= 2 we have the following equations :

$$\boxed{\left(\frac{\rho_A}{2R}\right)^2 = \frac{v^2+1}{v^2+4} \qquad \tan V = -1/v \qquad m = 2} \quad (4)$$

$$\boxed{\theta_A = \tfrac{1}{2}\arctan\tfrac{v}{2} - \arctan v} \quad (5)$$

For nephroid m= 1/2 so we have the following equations :

$$\boxed{\left(\frac{\rho_N}{R}\right)^2 = \frac{4(u^2+1)}{4u^2+1} \quad \tan V = -1/u \quad m' = 1/2 = 1/m} \quad (6)$$

$$\boxed{\theta_N = 2\arctan 2u - \arctan u} \quad (7)$$

We see that the range of the vector radius is in [2R, 4R] for the two expressions of ρ so the two curves are inside the circular annulus and we know that they are constantly orthogonal for any rotation around the common center O of the curves. If in the parametric equations of the astroid we use the change of variable v=2.u then :

$$\left(\frac{\rho_A}{2R}\right)^2 = \frac{v^2+1}{v^2+4} = \frac{4.u^2+1}{4.u^2+4} = \left[\frac{u^2+1}{u^2+\frac{1}{4}}\right]^{-1} = \left(\frac{R}{\rho_N}\right)^2 \quad (8)$$

The polar angle for the astroid is : $\theta_A = \frac{1}{2}\arctan\frac{v}{2} - \arctan v$ (9)
With v=2u we find : $\theta_A = \frac{1}{2}\arctan u - \arctan 2u$ (10)
for the nephroid the polar angle is : $\theta_N = 2\arctan 2u - \arctan u$ (11)

So we see that :

$$\boxed{\theta_N = -2.\theta_A} \quad (12)$$

5.2 Central inverse of the astroid (step 2)

We have proved on one hand that $\rho_A.\rho_N = 2.R^2$ for correponding points and on the other hand that there is an angular dilation which

transforms the inverse of the nephroid (so with same θ_N) in the astroid with $\theta_A = -\frac{1}{2}.\theta_N$.
In a similar way the central inverse of the astroid can be transformed into the Nephroid by angular dilation $\theta_N = -2.\theta_A$.
The four curves are inside the circular annulus (O, R$\rightarrow$ 2R).
The reciproqual polar of a curve is the inverse of the pedal and is also the antipedal of the inverse. There is a circular graph : initial curve $\rightarrow$ pedal $\rightarrow$ inverse $\rightarrow$ pedal $\rightarrow$ initial curve. A classical result - see [4] - is that for a Lame curve : (C_n) $x^n + y^n = a^n$ the reciproqual polar (for a circle centered at O) is another Lame curve :

$$x^{\frac{n}{n-1}} + y^{\frac{n}{n-1}} = a^{\frac{n}{n-1}} \qquad (13)$$

And the pedal (O at the center) of a Lame curves (C_n) is :

$$(\rho/a)^{\frac{n}{n-1}} = (\cos\theta)^{\frac{n}{n-1}} + (\sin\theta)^{\frac{n}{n-1}} \qquad (14)$$

For the astroid $n = \frac{2}{3}$ the reciproqual polar with $(\frac{n}{n-1} = -2)$ is :

$$x^{-2} + y^{-2} = a^{-2} \qquad \text{the Crosscurve} \qquad (15)$$

Using formula (14) with a= 2R the pedal of (C_{-2}) is :

$$(\frac{\rho}{2R})^{\frac{2}{3}} = (\cos\theta_A)^{\frac{2}{3}} + (\sin\theta_A)^{\frac{2}{3}} \qquad (16)$$

Which is the inverse of the astroid by the property of the reciproqual polar mentioned above. We use now the angular dilation $\theta_N = -2.\theta_A$ (step 1) to transform the central inverse of the astroid in the nephroid. This gives finally the central polar equation of the nephroid (3) :

$$\left(\frac{\rho}{2R}\right)^{2/3} = \left(\cos\frac{\theta_N}{2}\right)^{2/3} + \left(\sin\frac{\theta_N}{2}\right)^{2/3}$$

5.3 Parametric equations of Astroid, nephroid and their inverses

Equations of astroid t usual parameter are :

$$x_a = 3\cos t + \cos 3t = 4\cos^3 t$$

$$y_a = 3\sin t - \sin 3t = 4\sin^3 t$$

$$\rho^2 = 10 + 6\cos 4t = 16(\cos^6 t + \sin^6 t) \qquad ds = 6\sin 2t.dt$$

Equations of the central inverse $(O, 2.R^2)$ curve are :

$$X_{ia} = \frac{16\cos^3 t}{5 + 3\cos 4t}$$

$$Y_{ia} = \frac{16\sin^3 t}{5+3\cos 4t}$$

$$\rho^2 = \frac{32}{5+3\cos 4t} \qquad ds = \frac{24\sin 2t}{5+3\cos 4t}.dt$$

Equations of nephroid t usual parameter are :

$$x_c = 3\cos t + \cos 3t = 2(1+2\cos^2 t)\sin t$$

$$y_c = 3\sin t + \sin 3t = 4\cos^3 t$$

$$\rho^2 = 10 + 6\cos 2t \qquad ds = 6\sin t.dt$$

Equations of the inverse $(O, 8.R^2)$ curve are :

$$X_{id} = \frac{4\sin t(2+\cos 2t)}{5+3\cos 2t}$$

$$Y_{id} = \frac{8\cos^3 t}{5+3\cos 2t}$$

$$\rho^2 = \frac{8}{5+3\cos 2t} \qquad ds = \frac{12\cos t}{5+3\cos 2t}.dt$$

The central inverse of the astroid and the central inverse of nephroid are also ortho-curves for any rotation around the pole O.

6 Generalization of the angular dilation property to cycloidals

6.1 Angular homothetic of the central inverse of couple of orthocycloidals

The above computation for the couple astroid/nephroid can be generalized to any couple of orthocycloidals. We see that the range of the vector radius is in [1, m] or [R, m.R] for the two expressions of ρ so the two orthocycloidals are inside the circular annulus and we know that they are constantly orthogonal for any rotation around the common center O of the curves. The inversions we use in the paper exchange the two border-circles of the annulus. We use parameter $m = \frac{R}{R+2r} > 1$ with $r < 0$ for the hypocycloidal and $m' = \frac{1}{m}$ for the ortho-associated epicycloidal. In the parametric equations of the cycloidals we use the change of variable v=m.u then :

$$\left(\frac{\rho_1}{mR}\right)^2 = \frac{v^2+1}{v^2+m^2} = \frac{m^2.u^2+1}{m^2.u^2+m^2} = \left[\frac{u^2+1)}{u^2+\frac{1}{m^2}}\right]^{-1} = \left(\frac{R}{\rho_2}\right)^2 \tag{16}$$

The polar angles are for the hypocycloid 1 :

$$\theta_1 = \frac{1}{m} \arctan \frac{v}{m} - \arctan v \qquad (17)$$

we use v=m.u to transform the first hypocycloid angle and find :

$$\theta_1 = \frac{1}{m} \arctan u - \arctan m.u \qquad (18)$$

for the epicycloid 2 : $m' = \frac{1}{m}$, the polar angle is :

$$\theta_2 = m \arctan m.u - \arctan u \qquad (19)$$

So we see that :

$$\boxed{\theta_2 = -m.\theta_1} \qquad (20)$$

The similarity with couple astroid/nephroid permits only to define the transformation that applies the two orthocycloidals one on the other. If we begin with the hypocycloid 1 : find the central inverse $(O, m.R^2)$ then apply the angular dilation (20) we get the orthocycloidal : epicycloid 2. Conversely if we begin with epicycloid 2, inversion combined with angular dilation give hypocycloid 1. The central inverse of hypocycloid 1 and the central inverse of epicycloid 2 are also orthocurves for any rotation around the pole O.

6.2 Orthogonal trajectories of cycloidals in central polar coordinates

We examine orthogonal trajectories for rotations of cycloidals around the pole O center of the director circle. Using central parametric polar coordinates of Gomez Teixeira.

$$\left(\frac{\rho}{R}\right)^2 = \frac{u^2+1}{u^2+m^2} \quad m = \frac{R}{R+2r}$$

$$\theta = \frac{1}{m} \arctan \frac{u}{m} - \arctan u$$

We know that for cycloidals $\tan V = -\frac{1}{u}$ so at point of intesection $(\rho = \rho_\perp)$ of the orthogonal curve $\tan V_\perp = u$ so $\tan V . \tan V_\perp = -1$ for orthogonality. We have (setting R=1):

$$\tan V_\perp = \frac{\rho^2 . d\theta_\perp}{\rho . d\rho} \longrightarrow d\theta_\perp = u \frac{\rho . d\rho}{\rho^2} = \frac{(m^2-1)u^2 . du}{(u^2+m^2)(u^2+1)}$$

We find :

$$\theta_\perp = m.\arctan\frac{u}{m} - \arctan u \qquad \rho_\perp^2 = \frac{u^2+1}{u^2+m^2}$$

The expression of the angle $\theta_\perp$ can be transformed using the same changes of variable as above $v = m.u$:

$$\theta_\perp = -m.\Big[\frac{1}{m}\arctan\frac{v}{m} - \arctan v\Big] = -m.\theta$$

$$\rho_\perp^2 = \frac{v^2+1}{v^2+m^2} = \frac{m^2.u^2+1}{m^2.u^2+m^2} = \frac{u^2+\frac{1}{m^2}}{u^2+1} = \frac{u^2+m'^2}{u^2+1} = \frac{1}{\rho_2^2} \qquad m' = 1/m$$

It confirms the result found in section above on couple orthocycloidals and couple of inverse w.r.t. the center O.

7 Another example of orthocycloidals : Deltoid and Cardioid

The deltoid and the cardioid are a couple of orthocycloidals for m=3 an m'=1/3. By the same argument as before we have a square of curves using the two transformations : central inversion in ρ and angular dilation around O with parameter : m=3. So we set v=3.u.

$$\Big(\frac{\rho_1}{3R}\Big)^2 = \frac{v^2+1}{v^2+3^2} = \frac{3^2.u^2+1}{3^2.u^2+3^2} = \Big[\frac{u^2+1}{u^2+\frac{1}{3^2}}\Big]^{-1} = \Big(\frac{R}{\rho_2}\Big)^2$$

The polar angles are for the deltoid : $\theta_1 = \frac{1}{3}\arctan\frac{v}{3} - \arctan v$ (21)
we use v=3.u to transform the deltoid angle and find :

$$\theta_1 = \frac{1}{3}\arctan u - \arctan 3.u \qquad (22)$$

for the cardioid the polar angle is :

$$\theta_2 = 3\arctan 3.u - \arctan u \qquad (23)$$

So we see that :

$$\boxed{\theta_2 = -3.\theta_1} \qquad (24)$$

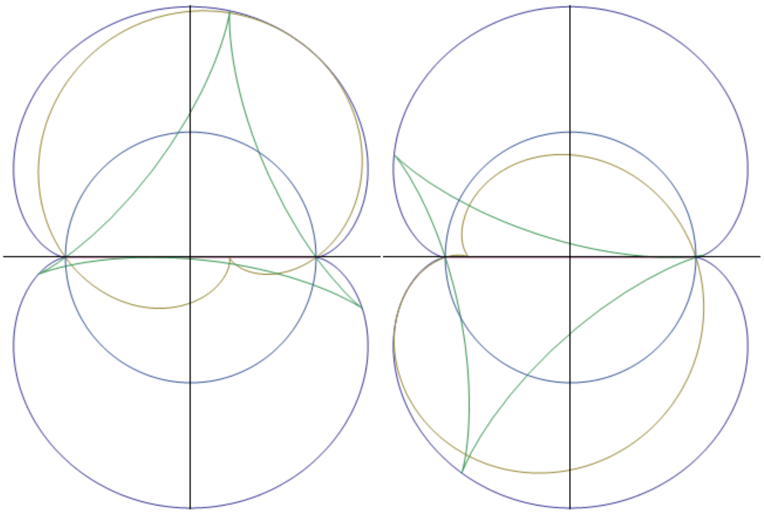

Figure 5: Deltoid and Cardioid as a couple of ortho-cycloidals.

7.1 Parametric equations of deltoid, cardioid and their inverses

Equations of the deltoid with t usual parameter are :

$$x_d = 2\cos t + \cos 2t = 2(1+\cos t)\cos t - 1$$
$$y_d = 2\sin t - \sin 2t = 2(1-\cos t)\sin t$$
$$\rho^2 = 5 + 4\cos 3t \qquad ds = 2\sin(3t/2)dt$$

Equations of the inverse $(O, 3.R^2)$ curve are :

$$X_{id} = \frac{3(2\cos t + \cos 2t)}{5+4\cos 3t}$$
$$Y_{id} = \frac{3(2\sin t - \sin 2t)}{5+4\cos 3t}$$
$$\rho^2 = \frac{9}{5+4\cos 3t} \qquad ds = \frac{12\sin(3t/2)}{5+4\cos 3t}.dt$$

Equations of cardioid with t usual parameter are :

$$x_c = 2\cos t + \cos 2t = 2(1+\cos t)\cos t - 1$$
$$y_c = 2\sin t + \sin 2t = 2(1+\cos t)\sin t$$
$$\rho^2 = 5 + 4\cos t \qquad ds = 2\cos(t/2)dt$$

Equations of the inverse $(O, 3.R^2)$ curve are :

$$X_{id} = \frac{3(2\cos t + \cos 2t)}{5 + 4\cos t}$$

$$Y_{id} = \frac{3(2\sin t + \sin 2t)}{5 + 4\cos t}$$

$$\rho^2 = \frac{9}{5 + 4\cos t} \qquad ds = \frac{12\cos t/2}{5 + 4\cos t}.dt$$

The central inverse of the deltoid and the central inverse of cardioid are also orthocurves for any rotation around the pole O.

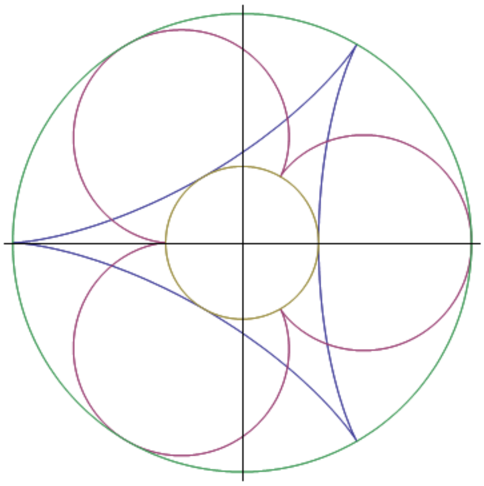

Figure 6: The Deltoid and central inverse (m=3)

8 Orthocycloidals as antipedals of rhodoneas : $\rho = \cos m.\theta$ and $\rho = \cos(\frac{\theta}{m})$

Antipedal of rhodoneas are cycloidals : hypocycloids when $m > 1$ and epicycloids when $m < 1$.
In [5] J. Lemaire presents a tangential generation of cycloidals in a simple manner. Given a fixed circle (O,R) and on it an origin of angles S. A first point P is at angle position t (the parameter of our curves) and second point Q is at angle position h.t (h real number). $h > 0$ for epicycloidals and $h < 0$ for hypocycloid (J. Lemaire uses the opposit values for h).The envelope of line PQ when t varies is a cycloidal. With this geometric tangential definition we easily find the equation of the pedal since :

$$\rho = R\cos\frac{h-1}{2}t \qquad \theta = \frac{h+1}{2}t$$

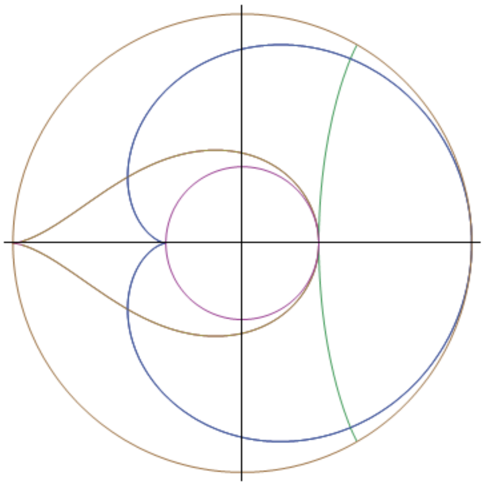

Figure 7: The Cardioid, its inverse and arch of deltoid (m=3)

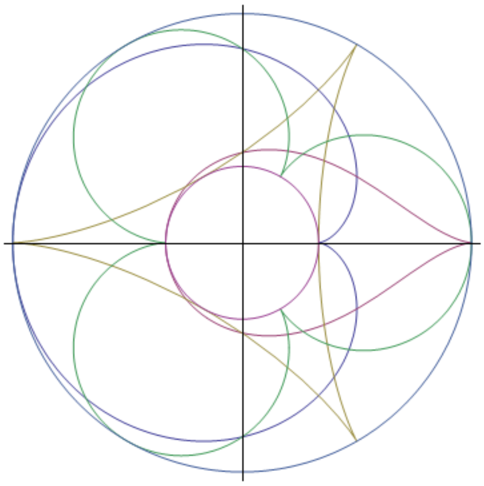

Figure 8: Deltoid, Cardioid and inverses (m=3)

$$\boxed{\rho = R\cos\frac{h-1}{h+1}t}$$

And antipedal of these rodhoneas are cycloidals (see PartXI).
$h < 0$ Hypocycloids, $h > 0$ Epicycloids, $h = 1$ a circle and $h = -1$ the 2-hypocycloid or a segment (degenerated). h= 2 : Cardioid, h=-2 Deltoid, h=3 : Nephroid, h=-3 Astroid, etc. If we change h in -h then the rodhonea is $\rho = R\cos\frac{h-1}{h+1}t = R\cos mt$ and becomes $\rho = R\cos\frac{h+1}{h-1}t = R\cos\frac{t}{m}$. For two orthocycloidals : $m = m$ and $m' = \frac{1}{m}$ or $m = \frac{h-1}{h+1}$ and $m' = \frac{h+1}{h-1}$. So parameter h is the same as the k used in the formulas for parametric cycloidals in section 3 above.

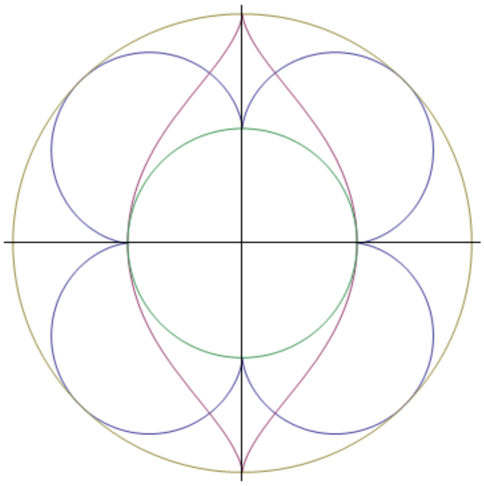

Figure 9: Orthocurves for rotation (transformed of astroid and deltoid by central inversion) m=2

References :

- [1] F. Gomez-Teixeira, Traite des courbes planes remarquables (Tomes 2 pp 273-274 3 pp 175, 180)
- [2] Robert C. Yates, A handbook on curves and their properties (1952). Ann Harbor MI - JW Edwards, p 153
- [3] J. D. Lawrence, A Catalog of Special Plane Curves, Dover, (1972), p 170
- [4] H. Brocard, T. Lemoine - Courbes geometriques remarquables Blanchard Paris 1967 (tome 2 p 58)
- [5] J.Lemaire, Hypocycloides et Epicycloides Librairie scientifique Blanchard (1967) Chapters 1-2. New edition of Vuibert (1929).
- [6] F. Morley, American Journal of Mathematics Vol. 13, No. 2 (Jan., 1891), pp. 179-184 This article is the 18^{th} on plane curves.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
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Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr