Tangential dual of Steiner-Habich Theorem circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line. - Part XX -

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Abstract

We present a dual of Steiner-Habich theorem : tangential generation of curves as envelope of a line attached to a curve rolling on a line. We apply this to the study of circular tractrix, its evolute and the pedal of this evolute. This gives a simple method to generate tangentially the circle as the roulette of a plane curve (a newtonian catenary) on a line.

1 A dual theorem of Steiner-Habich : tangential generation

The theorem of Steiner Habich (Part I) gives a simple relation between two planes curves :
If a curves (C) rolls without slipping on a base-line (D) a point P fixed to this curve traces the roulette (C'). If (C") is the pedal - from point P - of the curve (C) then the curve (C") is the wheel for the curve (C') w.r.t. the base-line. With initial conditions supposed to be verified.
This theorem is useful to study relations between plane curves. The theorem of Steiner Habich has a dual equivalent : instead of a point P we choose a given line (L) fixed to the curve (C) : if the curve (C) rolls on the base-line (D) then the line (L) will have for envelope the locus of the orthogonal projection of the CIR (= point of contact of (C) with (D)).
And we have the dual of Steiner-Habich theorem :
If a curves (C) rolls without slipping on a base-line (D) a line (L) fixed to this curve has for envelope the curve (C'). This curve (C') is also the roulette of the antipedal of the wheel for the ground (C') and base-line (L).

2 Glissettes of a plane curve

The glissette of a curve [2] in the plane is the locus of a point fixed to the curve when this one move and stays tangent to a line in base plane at a fixed point. If we choose x-axis as the line and O origin and if the curve is given in polar coordinates by (ρ, θ) then the glissette is (ρ, V) with V the angle between vector radius and the oriented tangent at current point M, so :

$$\tan V = \frac{\rho d\theta}{d\rho}$$

L. Braude recalls in [6] the following property (Aoust 1873, E.Duporcq 1901, Turrire 1909 see [5]) : the glissette of a point is the roulette on the normal (y-axis if the curve slide at O on x-axis) of the evolute for the same point.
An example is the glissette of an involute of the circle, O at the center, so O runs along a vertical line : the roulette of the center circle for y'y as base line. I we replace O by a point on the circle we get a cycloid, and by any point in the plane of the involute : a trochoid.
L. Braude chooses a tangential point of view : the envelope of a line (x-axis) fixed to gliding curve (C) $y = f(x)$ and uses the natural parameter $\frac{dy}{dx} = \tan\psi$, tangential equation of the envelope (E) is :

$$x.\sin\psi + y\cos\psi - y = 0$$

$$E \equiv x.\sin\psi + y\cos\psi - \int R\sin\psi.d\psi = 0$$

R is the radius of curvature. The tangential equation of the evolute of (E) is :

$$E_1 \equiv x.\cos\psi - y\sin\psi - R\sin\psi = 0$$

$$E_1 \equiv x.\cot\psi - y - R = 0$$

So the evolute (E_1) is the envelope of a parallele to the current tangent of (C) passing through the point A on Oy with OA = radius of curvature. As a consequence the evolute of the envelope (E) is the arcuide (see Part XIX) of the evolute of (C).
If instead of (C) $y = f(x)$ we take : $y = f(x) + c$ then the envelope of the x-axis is parallele curve to (C). But the glissettes of two parallele curves generate the same curve translated along y-axis. Further if we search the envelope of a line g_1 that cuts x-axis at point P and with angle α, then the envelope of line g_1 is a Koestlin transformation (x-axis, α).

3 Orthogonal trajectories of families of circles

The orthogonal trajectories of families of circles in the plane is a mean to find curve with similar properties. We need the parametric equation of the curve

$(x(t), y(t)$ on which moves the center C and a variable radius : $R(t)$ where M is placed. L. Ballif in [8] studies the $\perp$-trajectories of general families of circles and gives the general equation (θ is the azimuth of M on the circle):

$$X = x + R\cos\theta \qquad Y = y + R\sin\theta \qquad \tan\theta = \frac{dY}{dX}$$

$$\sin\theta.dx - \cos\theta.dy - R(t).d\theta = 0$$

he notes that dR is not in this equation.

$$\sin\theta\frac{dx}{R(t)} - \cos\theta\frac{dy}{R(t)} - d\theta = 0$$

He defines a kind of similarity between circles families such that :

$$\frac{dx}{R(t)} = \frac{dx'}{R'(t)} \qquad \frac{dy}{R(t)} = \frac{dy'}{R'(t)}$$

Which leads to a relation between arc length of $\perp$-trajectories :

$$\frac{ds^2}{R^2(t)} = \frac{ds'^2}{R'^2(t)}$$

L. Ballif notes that orthogonal trajectories of a family of circles can always come down to the ones of a family of constant radius circles.

3.1 Tractrices of a general curve

The Tractrix of a curve is the trajectory of the end of a rod of constant length b when the other end moves on the curve (C). This is a kinematic problem that depends on the position of the rod at the starting point as the initial condition. The problem was proposed by Claude Perrault around 1676 to Leibniz : find the curves solution of the special case when the curve C is a straight line. It has been solved by some mathematicians at the beginning of the new calculus invented by Leibniz and Newton. For some special curves (C), as the line or the circle the solution can be calculated by elementary functions. Since the tractrix needs an integration there are in general infinitely many tractrices of a given general plane curve.

3.2 Equitangential curves

For a given curve the definition of the equitangential is the locus of the point on the current tangent at distance a forward or backward if a positive direction is fixed on the curve. There are only two equitangential curves associated to a given curve in the plane. The equitangential curves are often special or limit solutions of the tractrix problem.

4 The circular tractrices

The circular tractrix is the trajectory of the free end M of a rod of lengh b when the other end A is forced to move on a circle of radius a. For b=a The curve is the tractrix spiral and was known by Huygens (1692), Varignon (1704) and the general circular tractrices have been studied by Bordoni (1820).
F. Morley in paper [3] of 1899 with a title about Amsler planimeter, he studies in fact the circular tractrices and gives a complete description of the general case of circular tractrix and their evolutes (in this paper we use Morley's notations). He presents the circular tractrices as the orthogonal trajectories of the circles of radius b with center on the circle (O, a) and the global invariance of the curve in the inversion $(O, a^2 - b^2)$ that transforms by inversion the two circles of the ring. This is equivalent to the tractrix problem if the circle has radius : a and the rod length : b. Three cases can be distinguished :

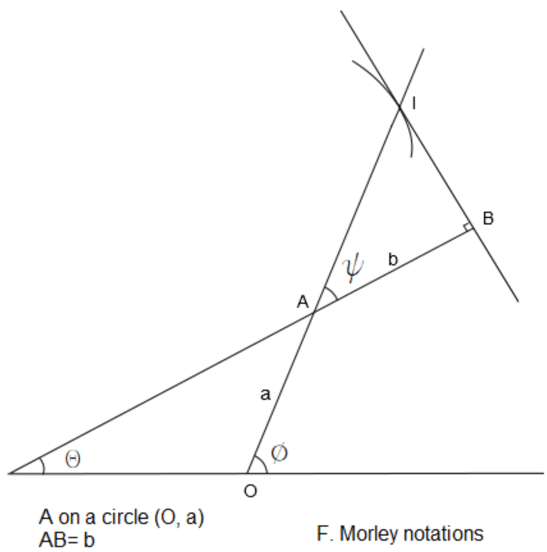

Figure 1: Morley's notations : A on a circle center O and rod AB=b.

4.1 Circular Tractrices for $b < a$

If $b < a$ then the circular tractrix has an asymptote circle with radius $\sqrt{a^2 - b^2}$.The curve is made of two pieces : one for an initial position of rod a inside the circle (O, a) and one for an initial position outside. These two curves are inverse for the inversion mentioned above that keeps the asymptote circle. There is a singular solution when the rod is (in initial position) tangent to the asymptote circle.

4.2 Circular Tractrices $b > a$:

If $b > a$ then the circular tractrix is made of successive arcs joigned by cusps. In general the curve, repeated ad infinitum, occupies all the space inside the two circles of the ring. But if the angle viewed from center of the fixed circle O between the two cusps is commensurable with $2.\pi$ so if $\sqrt{b^2 - a^2}/b = m/n$ with m,n $\in N$ and $m \cap n = 1$ then we can find an infinite number of closed curves than runs around O and come back to inital position just as for polygons or cycloidals.

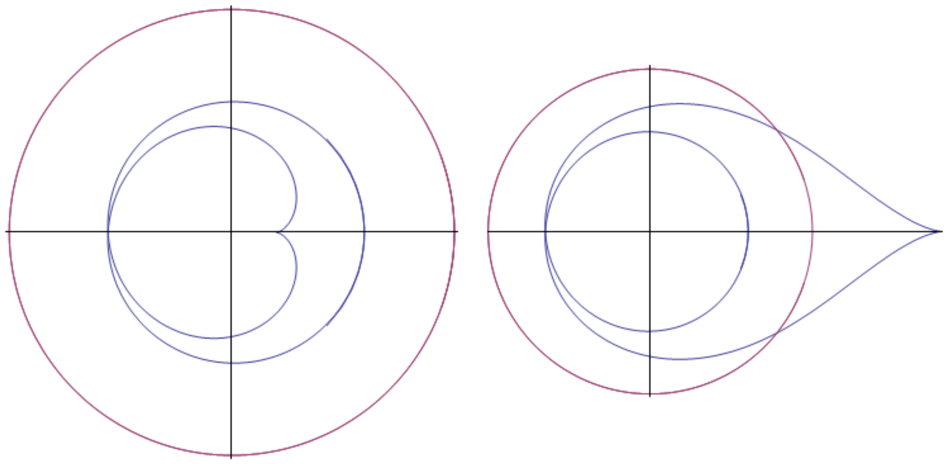

Figure 2: circular tractrices for $b = 2a/5$: the two curves are transformed by inversion from O.

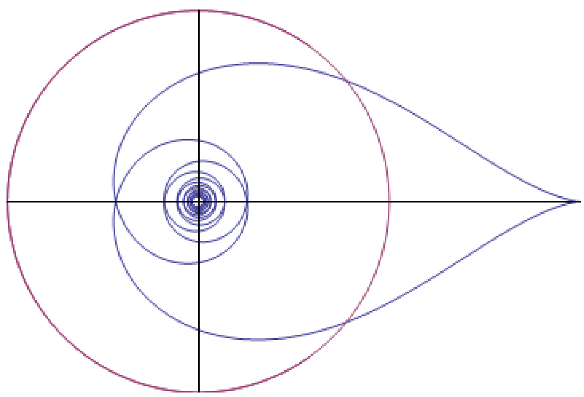

Figure 3: Tractrix spiral : $b = a$ $R_1 = 0$.

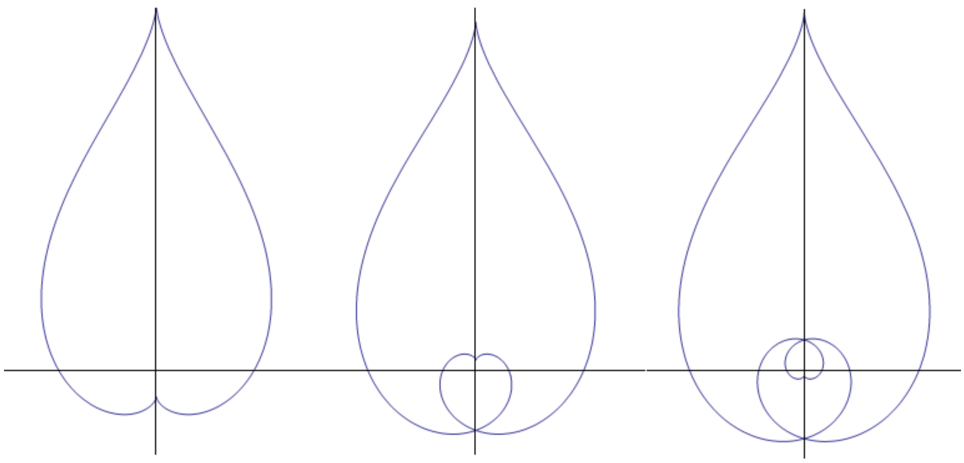

Figure 4: Closed circular tractrices in two pieces : 3 first cases of the serie.

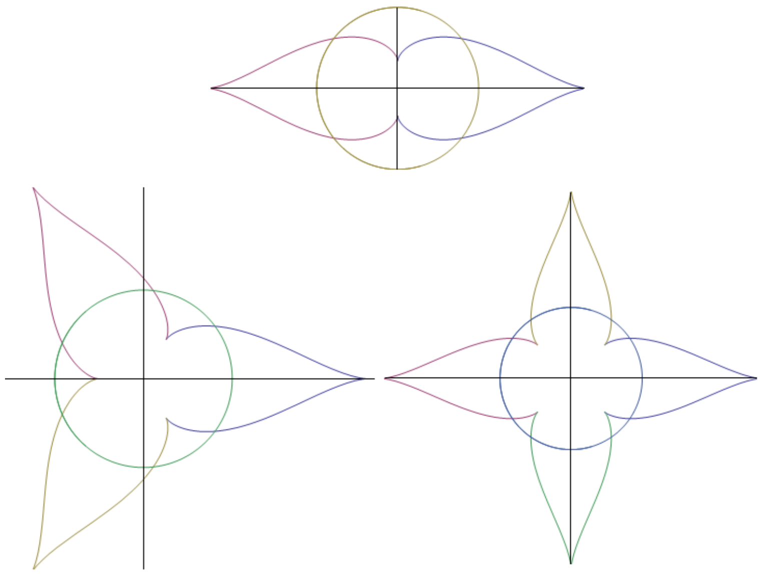

Figure 5: Closed circular tractrices.

4.3 Circular Tractrix for $b = a$: tractrix spiral

The intermediate case for $b = a$ is the well known tractrix spiral :

$$\rho = 2.a \cos u \qquad \theta = \tan u - u$$

The asymptote circle is reduced to a point. The circles in the rings pass through O so their transformed by inversion $(O, 4a^2)$ are tangent to a circle. The orthogonal trajectories of these tangents are the involutes of the circle, and so the tractrix spiral is the inverse of the involute of the circle as we

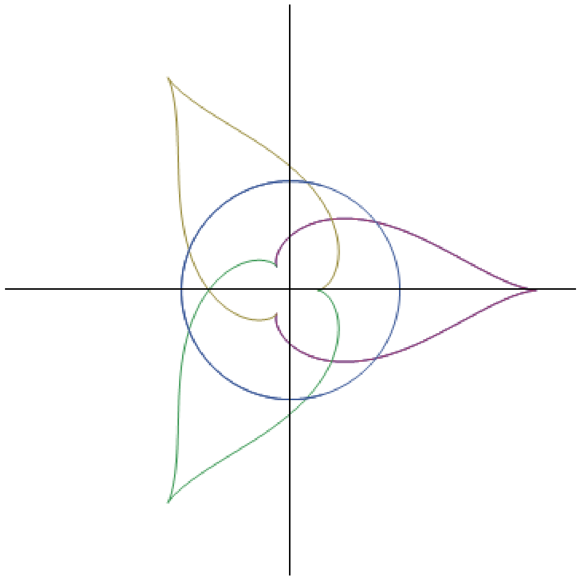

Figure 6: Closed 3-2 circular tractrices.

can directly see in the above parametric equations.

5 The circular tractrix as orthogonal trajectories of circles inscribed in a ring of two concentric circles

We have seen above that these orthogonal trajectories are circular tractrices and that those curves are globally invariant by inversion centered at O the common center. Since there are two kinds of circles tangent to the two base-circles of the ring we have two cases.

- The two points of tangential contact are on the same side of O, then the power inversion is positive. The point O is outside the $\perp$circles. Then the tractrices are those with $a > b$.

- The two points of tangential contact are on opposite sides of O, then the power of inversion is negative. The point O is inside the circles tangent to the ring. Then the tractrices are those with $a < b$. The curves are composed of finite identical arcs in form of S joined by cusps. In general the curves are not closed except for very special cases when :

$$\boxed{\lambda = \frac{\sqrt{b^2-a^2}}{b}} \qquad \text{is a rational number.}$$

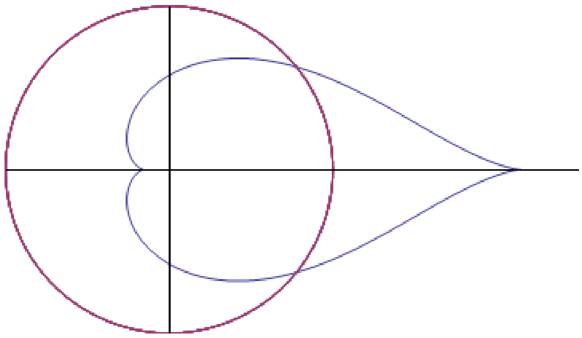

Figure 7: Circular tractrix fondamental curve for $b/a = 2/\sqrt{3}$ or $\lambda = 1/2$.

6 Cesaro equation of the circular tractrices

In [10] A. Kurnosenko gives the Cesaro intrinsic equation $f(s, R_c)$, in the formulas we use the form parameter $k = b/a$:

$$R_c(s,k,a) = \pm \frac{a.\sqrt{1-[k-(1+k)e^{-\frac{s}{a}}]^2}}{k-(1+k)e^{-\frac{s}{a}}} \qquad \text{with } k < -1$$

and the parametric representation with parameter $t = \tan\frac{\psi}{2}$:

$$\rho(t) = R\sqrt{1+2k\frac{1-t^2}{1+t^2}+k^2}$$

$$\theta_1(t) = \frac{2k}{\sqrt{k^2-1}}.\arctan\left(\sqrt{\frac{k-1}{k+1}}.t\right) - \arctan\left(\frac{k-1}{k+1}.t\right) - \arctan t \qquad 0 \le t \le \infty$$

$$\theta_2(t) = t - \arctan t \qquad \text{(k=1 Tractrix spiral)} \qquad 0 \le t \le \infty$$

$$\theta_{3,5}(t) = \frac{2k}{\sqrt{1-k^2}}.\arg\tanh\left(\sqrt{\frac{1-k}{1+k}}.t\right) + \arctan\left(\frac{1-k}{1+k}.t\right) - \arctan t =$$

$$= \frac{k}{\sqrt{1-k^2}}.\log\frac{\sqrt{1+k}+t.\sqrt{1-k}}{\sqrt{1+k}-t.\sqrt{1+k}} - \arctan\frac{2k.t}{1+k+(1-k)t^2} \qquad 0 \le t \le \sqrt{\frac{1+k}{1-k}}$$

F. Morley and A. Kurnosenko show that the two classes of circular tractrices correspond to two kinds of tangency for the circles in the ring (R_1, R_2) : those around O and the others. The case when the inside circle is a point is the intermediate case of spiral tractrix (a=b). Then the small circle R_1 is externally tangent the $\perp$-circles. For these circles the orthogonal trajectories correspond to $b < a$. Inversion is positive $R_1.R_2 = a^2 - b^2 > 0$. And the circular tractrix is made of two parts transformed by central inversion : one inside, one outside the asymptotic inversion circle.
If the two circles of the ring have opposite direction then the circles are internally tangent and the point O is inside the $\perp$-circles and we get the

other class when $a < b$. Inversion is negative $= R_1.R_2 = a^2 - b^2 < 0$.
The problem could better be examined in Laguerre geometry where circles have a direction and radii have a sign.

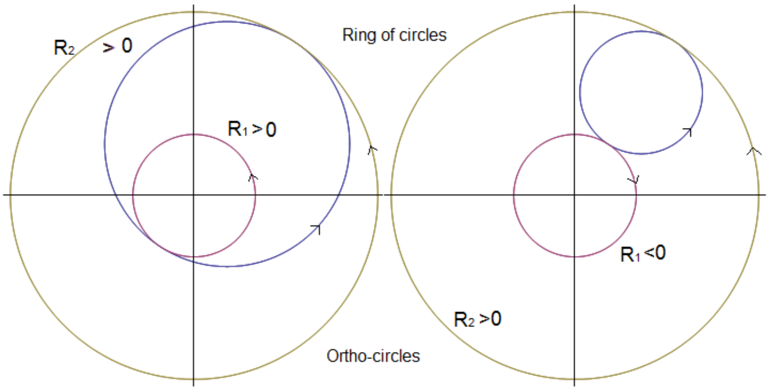

Figure 8: The two kinds of tangent circles inside a ring.

7 The problem of Catalan (1856). The circle as roulette of evolute of circular tractrices :

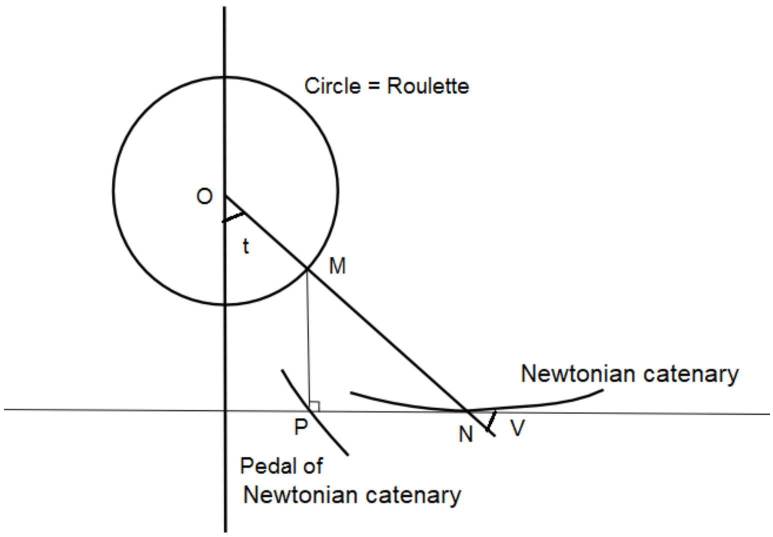

Figure 9: Roulette of a newtonian catenary is a circle.

In a paper of 1856 [1], E. Catalan looks for the curve (R) that, rolling on a given fixed curve the base : (B), generates as the roulette of a fixed point another given curve (C) in the plane. He applies this to the special case when (B) is a line and (C) a circle of radius a. The solution (R) depends on the respective positions of the line and the circle. Distance from center of the circle and line (B) is b. We already know two solution 1- if the line (B) is the diameter (b=0) of the circle (C) the curves (R) is reduced to a point at distance a : this is the definition of a circle centered on the base line, 2- if the line (B) is tangent (b=a) to the circle (B) then the curve (R) is Catalan's curve $\rho = \frac{2a}{1-\theta^2}$ (Catalan's curve see Part VI). There are two cases for the general solution $a > b$ and $a < b$. The latter is easier since the circle has no intersection with the base line and we will only consider this case.

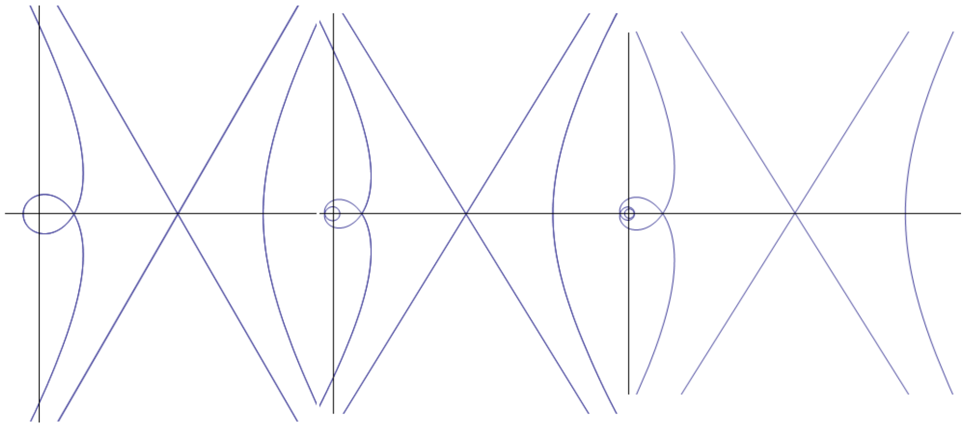

Figure 10: Algebraic newtonian Catenaries : first 3 cases : $\frac{b}{a} = \frac{n}{\sqrt{n^2-m^2}}$

7.1 The case $b > a$:

The base-line has no real intersection with the circle. E. Catalan gives in polar coordinates the equation of the curve (R) :

$$\rho = \frac{\frac{b^2-a^2}{a}}{1 \pm \frac{b}{a} \cos \frac{\sqrt{b^2-a^2}}{b}\theta}$$

We shall call these curves : newtonian catenaries.
They are a subclass of curves called polygasteroides (see mathcurve and Charles Laboulaye Traite de cinematique 1849) defined as transformed of conics (pole at a focus) by an angular dilatation $\theta \longrightarrow \lambda\theta$. In our case $e = b/a > 1$ the polygasteroide is the transformed of a hyperbola. When the ratio for the angular dilatation : $\theta \rightarrow \lambda\theta$, the polar angle is multipled by $\lambda = \frac{\sqrt{b^2-a^2}}{b}$ then the curves are the newtonian catenaries and there is a relation between λ and e of the general polar equation of polygasteroides :

$$\rho = \frac{A}{1 \pm e \cos \lambda.\theta}$$

The newtonian catenaries have two asymptotes are composed of two parts, one arc is like an hyperbola for the sign $-$ and the two other arcs for the sign $+$ are symmetric w.r.t. to xx' with a rotating part around the pole, sometimes with finite number of loops.

In general these curves are not closed and not algebraic except in special cases with the closing condition :

$$\lambda = \frac{\sqrt{b^2 - a^2}}{b} \text{ is a rational number } = \frac{m}{n}$$

So we have with this condition :

$$a = b\frac{\sqrt{n^2 - m^2}}{n}$$

Then the curves have the following polar equation :

$$\rho(t) = \frac{1}{1 \pm \frac{n}{\sqrt{n^2 - m^2}} \cos \frac{m}{n}\theta}$$

With the constraints : $m > n$, m, n $\in N*$ and $n \cap m = 1$ to get real curves. If we fix m=1 and if n is any integer we get the first serie corresponding to Euler-Serret curves with same arc length as the circle (see Part II) :

$$\rho(t) = \frac{1}{1 \pm \frac{n}{\sqrt{n^2 - 1}} \cos \frac{\theta}{n}}$$

Another definition of Newtonian catenary uses the properties of rolling curves about two poles O and O' at distance d. It is well know in cinematic (see Part I or mathcurve) that if a wheel C_1 rotates around O then we can find a second wheel C_2 turning around O' in such a way that the wheels roll one on the other without slipping and arc length are equals. If the wheel C_1 is a straight line with polar equation $\rho = a/\cos\theta$ then the other wheel is a newtonian catenary if a=b then the catenary is Catalan's curves. All these wheels have the same arc length as the line : $s = a\tan\theta$ or the standard catenary $s = a\sinh\frac{x}{a}$. We apply this at section 11.

8 Evolutes of circular tractrix and Euler-Serret curves

In his paper [3] of 1899 gives a solution of the same classical problem : to look for the curves in the plane the roulette of which is a circle when rolling on a fixed line (L). The solutions are the evolutes of the circular tractrices and, by Steiner-Habich theorem, the pedals of these evolutes are the wheels for the circles w.r.t. the same line (L). These wheels are the Euler-Serret curves we have studied in part II : plane curves with same arc length as the circle.

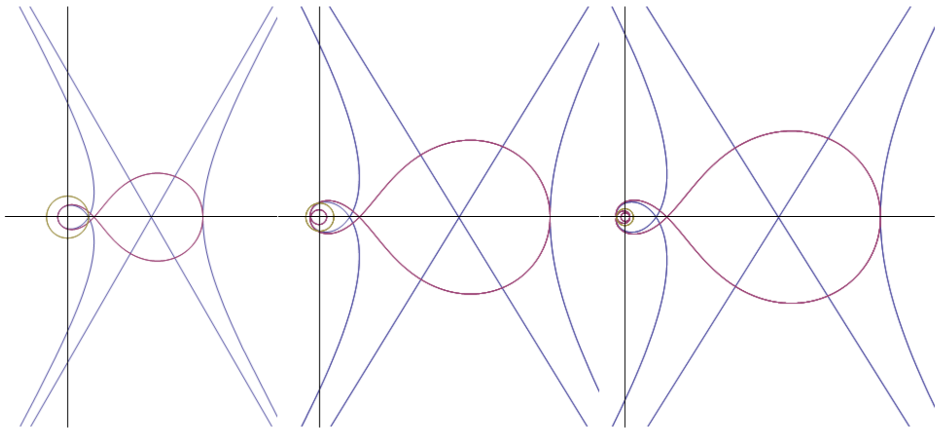

Figure 11: Newtonian catenaries and their pedals : 3 first cases the pedal is a wheel for the circle ground.

9 Catenary in a special case of newtonian potential

F. Morley gives also an important property of these curves : they are Catenaries for a newtonian central force in $1/r^2$. The radius vector of the catenary is $r = a + b/\cos\psi$ and the radius of the pedal is $p = b + a\cos\psi$ so $(r-a)(p-b) = ab$, equation equivalent to :

$$\boxed{\frac{a}{r} + \frac{b}{p} = 1}$$

And the arc length $s = b\tan\psi$. F. Morley uses the mechanichal properties of weighted chains in a central force field. If ψ is the angle between vector radius and tangent ($\cos\psi = dr/ds$), F the central force, T the tension in the chain, then the equilibrium of an element of arc (see [13]) is :

$$dT + F ds\cos\psi = 0 = \frac{dT}{dr} + F \qquad (1) \qquad T.p = A \qquad \text{(2) Moment Eq.}$$

The above pedal equation allows to prove that newtonian catenaries are the profile curves of a chain hanging between two points for a newtonian central force in $1/r^2$. The tension $T \propto 1/p$ and force $F \propto D_r T$. For Catalan's curves : $F \propto D_r(1 - a/r)T \propto 1/r^2$. It is the Newton inverse square law. For a catenary we have the following equations (from F. Morley [3]) :

$$r = a + b/\cos\psi \qquad p = b + a\cos\psi \qquad s = b\tan\psi$$

$$d\theta = \frac{b.d\psi}{b + a\cos\psi} \qquad \frac{\theta}{2b} = \frac{1}{\sqrt{b^2 - a^2}}\arctan\left[\sqrt{\frac{b-a}{b+a}}\tan\frac{\psi}{2}\right]$$

$$\lambda = \frac{\sqrt{b^2 - a^2}}{b} \qquad \tan\left(\lambda.\frac{\theta}{2}\right) = \sqrt{\frac{b-a}{b+a}}\tan\frac{\psi}{2}$$

$$\cos\lambda\theta = \frac{a + b\cos\psi}{b + a\cos\psi}$$

The parametric polar equation :

$$r = \frac{a^2 - b^2}{a - b\cos\lambda.\theta} = \frac{\frac{a^2-b^2}{a}}{1 - \frac{b}{a}\cos\lambda\theta}$$

$$\sin\lambda\theta = \frac{\sqrt{b^2 - a^2}\sin\psi}{b + a\cos\psi} = \frac{\lambda.s}{r}$$

$$\boxed{s = r.\frac{\sin\lambda\theta}{\lambda} = b\tan\psi}$$

10 Wheels for a catenary base line parallele to the directrix

Using Gregory's transformation, we can find the equations in parametric polar coordinates of the wheels corresponding to a catenary ground $y = \cosh x$ for base lines parallele to the directrix. We must consider the curve as double completing the usual one with its symetric w.r.t. the directrix. We already know two special cases when the base line is the directrix : the wheel is the line $\rho = 1/\cos\theta$ and when the base line is one of the the vertex tangents then the wheel is Catalan's curve $\rho = 1/(1 - \theta^2)$.

For the general case the catenary is $y = \cosh x + d$ so the above cases correspond to $d = 0$ and $d = \pm 1$. We distinguish two cases for $|d| < 1$ and $|d| > 1$ since the integral separates them (if d¿1 the base line cuts the catenary). And we have :

$$\rho = y = \cosh x \qquad \theta = \int \frac{dx}{y} = \int \frac{dx}{d + \cosh x}$$

So it is possible to find by one quadrature the parmetric polar equations $[\rho(t), \theta(t)]$ of newtonian catenaries but we will take another equivalent path to find these equations in next section.

11 Wheels corresponding to a straight line base line rolling one on the other around 2 poles

As we have seen in Part I, 2 wheels for the same ground and parallele base lines at distance d are rotating curves about 2 poles at distance d in the plane. Since the line $\rho = b/\cos\theta$ is one of these wheels (when the line is the

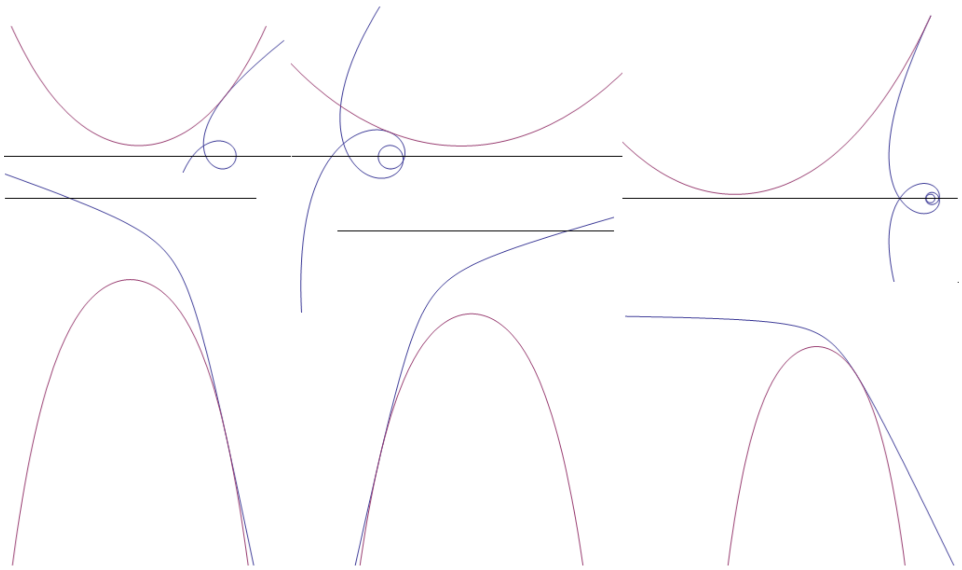

Figure 12: Wheels for catenary ground and line parallele to the base : 3 first cases with rational λ. Two branches : hyperbolic and with loops.

base of the standard catenary) all the newtonian catenaries are wheels for a parallele to the base line or a rolling curve on a line considered as one of the 2 wheels. We use this property to look for the polar parametric equations of the newtonian catenaries. Poles (O, O',d) and the (C1) turning around O

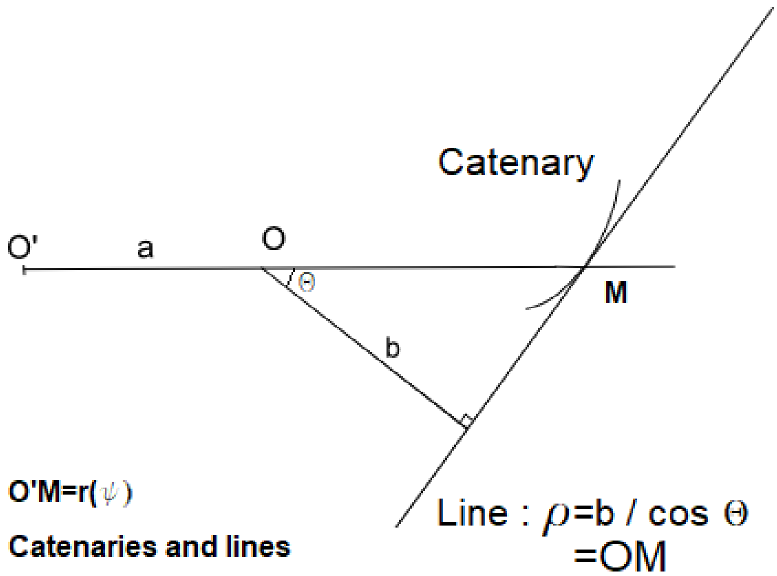

Figure 13: Wheels for lines and catenaries

is the line $\rho(\theta) = b/\cos\theta$. The condition for the curve (C2) to roll on (C1) are :

$$r(\psi) = \rho + a \text{ and } \qquad \rho(\theta).d\theta = r(\psi).d\psi$$

So we get :

$$d\psi = \frac{\rho.d\theta}{a+\rho} \quad \text{and} \quad d\psi = \frac{b.d\theta}{b + a\cos\theta}$$

$$\psi = \int \frac{b.d\theta}{b+a\cos\theta} = \frac{2b}{\sqrt{b^2-a^2}} \arctan\sqrt{\frac{b-a}{b+a}}\tan\frac{\theta}{2}$$

We obtain :

$$\tan\left(\frac{\sqrt{b^2-a^2}}{b}\frac{\psi}{2}\right) = \sqrt{\frac{b-a}{b+a}}\tan\frac{\theta}{2}$$

We set $\lambda = \frac{\sqrt{b^2-a^2}}{b}$ Or :

$$\tan\frac{\theta}{2} = \sqrt{\frac{b+a}{b-a}}\tan\frac{\sqrt{b^2-a^2}}{b}\frac{\psi}{2}$$

$$r(\theta) = a + \rho(\theta) = a + \frac{b}{\cos\theta} = a + b\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$$

Replacing $\tan\frac{\theta}{2}$ by its value funcion of $\tan\frac{\psi}{2}$ above, we have :

$$r(\psi] = \frac{a(1-\tan^2\frac{\theta}{2}) + b(1+\tan^2\frac{\theta}{2})}{1-\tan^2\frac{\theta}{2}}$$

$$r(\psi] = \frac{(b+a)+(b-a)\frac{b+a}{b-a}\tan^2\frac{\lambda\psi}{2}}{1-\frac{b+a}{b-a}\tan^2\frac{\lambda\psi}{2}}$$

$$r(\psi) = \frac{(b^2-a^2)[1+\tan^2\frac{\lambda\psi}{2}]}{(b-a)-(b+a)\tan^2\frac{\lambda\psi}{2}}$$

So :

$$r(\psi) = \frac{b^2-a^2}{(b-a)\cos^2\frac{\lambda\psi}{2} - (b+a)\sin^2\frac{\lambda\psi}{2}}$$

And finally :

$$\boxed{r(\psi) = \frac{\frac{b^2-a^2}{a}}{-1+\frac{b}{a}\cos\lambda\psi}}$$

with $\lambda = \frac{\sqrt{b^2-a^2}}{b}$.That is the formula of Catalan (1856) and F. Morley (1899), the transformed of conics by angular dilatation λ.

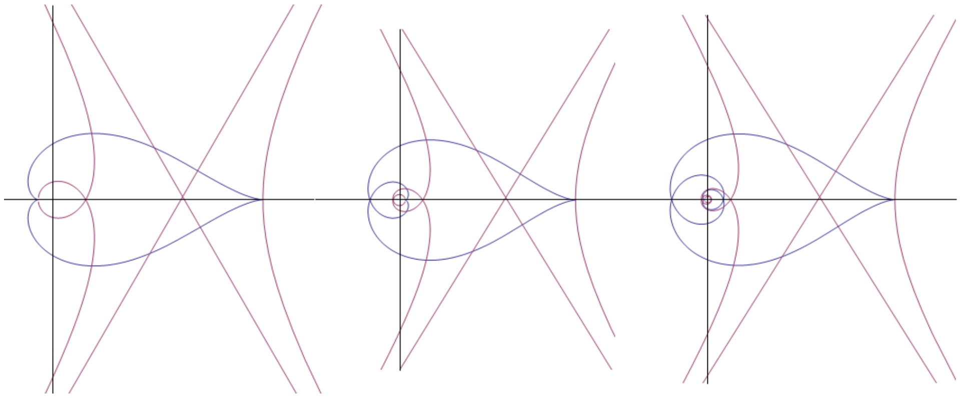

Figure 14: Catalan curves as evolutes of circular tractrices : 3 first cases

11.1 Evolute for $a > b$

The first curve (for $\lambda = 1/2$) is a circular quartic. The others are algebraic -if λ is rational - but of higher order. This figures show that the newtonian catenaries have a similar form as mentioned by F. Morley : "the general case can be grasped from this by adding more loops round the center, and by rotating the whole about the centre through equal angles".

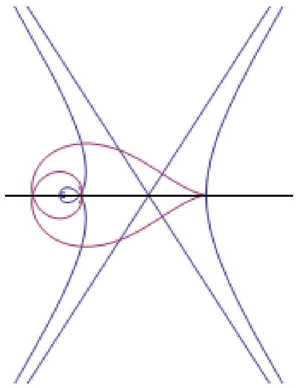

Figure 15: Tractrix spiral : $b = a$ and its evolute Catalan's curve $\rho = \frac{2b}{1-\theta^2}$

12 Circle as roulette of newtonian catenary

Newtonian Catenaries are the solution of the problem studied by E. Catalan in 1856 [1]. To find the curves that rolling on a line generate a circle as the roulette of pole O. E. Catalan gives the solution in polar coordinates, pole at O. F. Morley presents a new solution using the glissettes of circular tractrices. Since these curves are involutes of the newtonian catenaries (or Catalan curves) the locus of the pole when the catenary rolls on a line are circles. For the special case of tractrix spiral the catenary is the special Catalan's curve $\rho = 1/(1-\theta^2)$ the corresponding circle is tangent to the y axis at origin.

13 Circle as envelope of the parallele to the base of a catenary

When an ordinary catenary $y = \pm \cosh x$ rolls on a line the envelope of the base is a fixed point (just fix the line in the ground-wheel : catenary/line $\rho = 1/\cos\theta$), so the envelope of a parallele to the base is a parallele curve of a point : a circle. We must consider the catenary as a double curve : the symmetric catenary w.r.t. the directrix. Using the classic result in part I, if a newtonian catenary rolls on a line the roulette of the pole O is a circle in the fixed plane. The first piece upper part of the catenary generates the lower half circle, the lower catenary generates the upper half circle. The two parts are necessary to generate the complete circle.

14 Rolling parabola : focus generate upper catenary, directrix envelopes the lower catenary

The roulette of the focus of a parabola is a catenary when the base is a line. A defining property of the parabola is that the symmetric of the focus w.r.t. the current tangent is on the directrix.
When this parabola rolls on a line (base) the directrix has for envelope the catenary symmetric of the locus of F w.r.t. the base line. And which can also be generated by the focus F of the same parabola rolling under the base line (see an animation on mathcurve [14] at the page on Roulette de Delaunay).
These properties reflect the natural proximity of the circle, the standard catenary and the newtonian ones which have all the same arc length. And we propose a tangential definition of the circle with a curve and a line : a circle is the envelope of a line parallele to the base of a standard catenary when the latter rolls on a fixed line in the plane.
This kind of definition can be easily generalized to any curve in the plane if we use Gregory's transformation and dual Steiner Habich theorem.

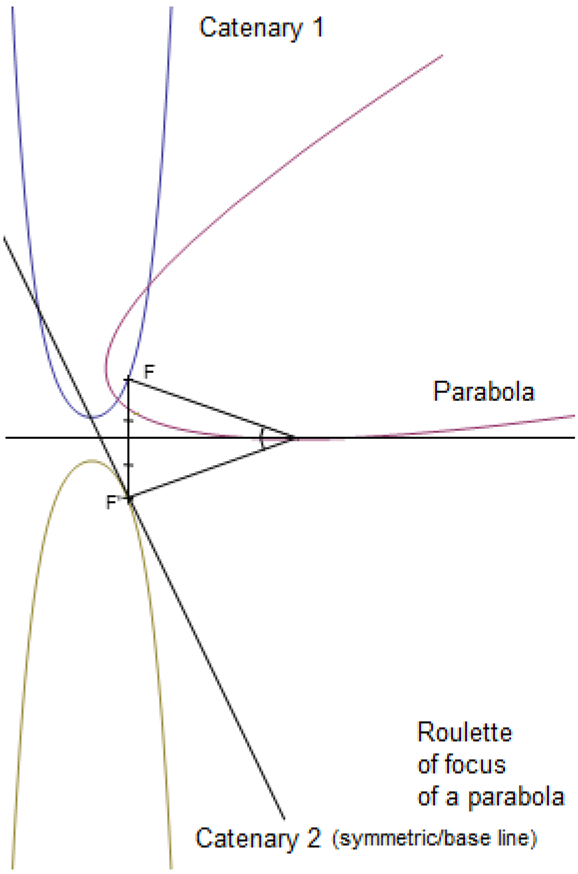

Figure 16: Roulette of a parabola : the focus is on the upper catenary, directrix is tangent to the lower one.

15 Trios of curves : Circular tractrices, newtonian catenaries and Euler-serret curves

To summarize we have seen that :
1 - The glissettes on the x-axis at origin for the center O of circular tractrices are circles.
2 - The roulettes on the y-axis for the center O newtonian catenaries (or Catalan curves) are the same circle.
3 - The newtonian catenaries are the wheels of the standard catenary $y = d \pm \cosh x$ as the ground.
4 - The evolute of a circular tractrix is a newtonian catenary.
5 - The pedal of these newtonian catenaries w.r.t. O are the Euler-Serret curves studied in Part II and by Steiner-Habich theorem are the wheels for a circular ground if the pole O runs on y-axis.

In Part II we have looked for closed and algebric curves and this happens for special values of the ratio possible only for $a > b$:

$$\boxed{\frac{\sqrt{b^2-a^2}}{b}}$$

is a rational number : $= \frac{m}{n}$ with m, n $\in N$ and $m \cup n = 1$. In this case the trio of curves : Circular tractrix (closed), its evolute : newtonian catenary (with real points at infinity) and the pedal from O (closed no real point at infinity) of this evolute are algebraic curves. The first cases m=1, n=2, 3 and 4 - with only a few loops - of these interesting curves are in illustrations 9-10-11. But there are many other cases.

16 Generalizations of the tractrix

We have seen that the circular tractrix is the orthogonal trajectories of a family of circles in the plane situated in a corona between two concentric circles. This method can be generalized to generate curves as orthogonal trajectories of the family of circles tangents to two lines or circles in the plane (families of circles in the plane have two envelope curves). E. Turriere in [7] studies this problem for circles tangent to two line. The solutions are easily obtained because two simple transformations keeps globally the given circles :
- an homotety from the point O of intersection of the given lines,
- an inversion from the point O of intersection of the given lines,
The equations of the circles are :

$$r^2 - 2\lambda.r\cos\theta + \lambda^2\cos^2\alpha = 0$$

Th differential equation of these circles (λ=parameter) :

$$\boxed{\left(\frac{dr}{r.d\theta}\right)^2 = \frac{\sin^2\theta}{\cos^2\theta - \cos^2\alpha}}$$

Then replace $\frac{-r.d\theta}{dr}$ by $\frac{dr}{r.d\theta}$ to find :

$$\boxed{\left(\frac{dr}{r.d\theta}\right)^2 = \frac{\cos^2\theta - \cos^2\alpha}{\sin^2\theta}}$$

The solution is :

$$\log r = \pm \int \frac{(\sqrt{\cos^2\theta - \cos^2\alpha}}{\sin\theta} d\theta + C$$

He uses an auxiliary parameter :

$$\cos\theta = \cos\alpha.\cosh u$$

So :

$$\log r = \int \frac{\cos^2\alpha.\sinh^2 u}{\cos^2\alpha.\cosh^2 u - 1} du = u + 2\int \frac{\sin^2\alpha.du}{\cos^2\alpha(\cosh 2u+1)-2}$$

Or :

$$\boxed{r = C.e^u.[\frac{e^{2u}-\tan^2(\frac{\pi}{4}-\frac{\alpha}{2})}{e^{2u}-\cot^2(\frac{\pi}{4}-\frac{\alpha}{2})}]^{\frac{\sin\alpha}{2}}}$$

This paper owes much to Emile Catalan 1856 [1] and Franck Morley 1899 [3].

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This article is the 20^{th} on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.

Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
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Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
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Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr