# Equality of arc length of the parabola and the Archimede spiral. A historical tale of a question that raised at the beginning of the calculus (1643-1668) Hobbes Roberval Mersenne Torricelli Fermat Pascal and J. Gregory. - Part XXII - 

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#### Abstract

We go through a short period of history of the beginning caculus (1643-1668) when two classical curves : the parabola and the Archimede spiral, were implicated. Archimede had settled some important ideas, first steps to calculus and applyied them to the parabola and the spiral for evaluating areas. In the 17th century an analogy between these two curves was of interest, not only for area but also for comparing the arc lengths and many geometers tried to prove or to generalize this property. All this can be easily explained in the final notation form of calculus in the Leibnizian manner and the use of polar coordinates.


## 1 Two curves coming from antiquity : Parabola and Archimede spiral

In the antiquity mathematics, the curves cut on a cone by a plane were classical objects of study and Apollonius had given some properties of these conics and a classification in the three types: Ellipses, including the circle, Parabola and Hyperbolas. Some other curves were described by compound motions like Hippias quadratrix or Achimede Spiral. This curve is generated by a right line in uniform rotation around a point O and a uniform move along the line -see (1).
Archimede had given some results on the construction of tangents or the area inside the first tour of the spiral : one third of the circle centered at O with radius $2 \pi . a$ if $\rho=a . \theta$ is the polar equation of the spiral. It is the area wiped by the radius vector from $t=0$ to $t=2 . \pi$.
In Seventeenth century the reading of the work of Archimede was an incitation to generalise his ideas and methods. It contains important means to calculate length, area and volumes, particularly the lenghth or area of circle, Parabola and also of his spiral.

This was a source of inspiration for mathematicians to calculate area limited by lines or plane curves and volumes limited by planes and simple surfaces. Kepler, Galileo, Cavalieri, Gregory of St Vincent, Guldin and others tried to improve Archimede's methods and it was the beginning of the story of Calculus carried to an efficient mean by Newton and Leibniz later about 1680-90. Cavalieri in Geometria indivisibilibus 1635 (1) used his new methods to evaluate the area inside the first turn of the Archimede spiral and reduced to the area limited by lines and a parabola.

The equation of these curves in orthogonal coordinates and polar coordinates are :

$$
y^{2}=2 . a . x \quad \text { Parabola }
$$

$$
\rho=a . \theta \quad \text { archimede spiral. }
$$



Archimed spiral : $\rho=\mathbf{a} \cdot \vartheta$
Figure 1: Archimede's method for inside area of his spiral


Figure 2: Cavalieri's method for outer Area of 1st turn of Archimede spiral

## 2 Mersenne and his friends

Marin Mersenne (1588-1648) joined Minim Friars in 1611 and was from 1618 to his death lecturer at the "Couvent des minimes" of the Place royale (today "place des Vosges") in Paris. He travelled in different countries : Low countries (1629-1630), East of France (1639), Provence and Italia (1644-1645), provinces of the west et du South-west (1646-1647). His correspondance, in which he suggest many interessant questions (the cycloid, "la roulette" for example), that is an excellent stimulation for activity of savants and help publications of new ideas and scientific works.
SA Shirali indicates in (3) :"By about 1620 he had also decided for himself that alongside religious studies he would devote his time to science and mathematics. This interest in combination with his meetings with scholars soon gave rise to an extraordinary and unprecedented tradition, in which he began to keep contacts with a number of scientists and mathematicians (including some whom he never met) : Descartes, Fermat, Etienne Pascal and Blaise Pascal, Roberval, and many others. He set up meetings with them in which they would discuss their work.
This informal academy began to be known as Academie Mersenne. At one such meeting Mersenne persuaded Roberval to work on the cycloid, and this brought forth rich dividends" ... "Mersenne's letters run into thousands of pages. After his death in 1648, letters were found in his cell from seventy eight different correspondents in several countries. Read chronologically, they offer a very insightful glimpse at how mathematics and mechanics were evolving during this period of ferment. It is no exaggeration to say that he was the creator of a scientific academy that stretched across the length and breadth of Europe."

After 1635, he carried out his project to organize collective work and animated the academia parisensis. This "confrerie" became an institution and later Colbert en 1666 created the French science academy.
Mersenne awes his reputation to his role of facilitator of scientific life in the $1630-1640$ years. And Hobbes described him as "the one around who turned as around an axis each star of the science each on his own orbite".

## 3 A meeting in the cloister of convent "les Minimes" Paris in December 1642

In the winter 1642-43 Mersenne encountered Hobbes and Roberval, who had recently found the area between an arch of the cycloid and its base-line, at his monastery "des Minimes" in Paris. And they talked of the arc length of the parabola and spiral.
In fact the analogy Parabola-Spiral is composed of two parts :
1 - A relation between areas under the parabola and the sector between two vectors radii and the spiral with factor $A_{s}=\frac{1}{2} \cdot A_{p}$. In a way this was proved by Archimede and Cavalieri.

2 - An equality of arc length between two peer points on the parabola and on the spiral.

Hobbes was in Paris during the civil war from 1640 until 1651, because he feared for his safety. Shortly after completing The Elements of Law, he fled to Paris, where he rejoined Mersennes circle and made contact with other exiles from England.
In the "Cambridge companion to Hobbes 1996" we read under the pen of Hardy Grant (6) "...At some time during the winter 1642-43 Hobbes paid a visit to Marin Mersenne at the latter's Minime convent in Paris. Present also were Roberval and an unnamed fourth person. Mersenne, intellectuel gadfly and go-between extraordinaire was like Hobbes only an amateur in geometry, but Roberval would later hold a chair in mathematics at the College Royal. By his own account, which Mersenne corroborated, Hobbes chalked on the wall of the convent an argument for the equality of arcs of the two curves, spiral and parabola, have equal lengths. While not of course a rectification of either curve, this result if valid, would have been of much interest and importance in its own right."
Hobbes explains (translated from latin) :"Being with Mersenne and Roberval in the cloister of the convent, I drew a figure on the wall, and Mr Roberval, perceiving the deduction I made, told me that since the motions which make the parabolical line, are one uniform, the other accelerated, the motions that makes the spiral must be also; which I presently acknowledge; and he the next day, from this very method, brought to Mersenne the demonstration of their equality."
Hobbes was not so good mathematician like he pretended "and it is true he had no remarkable influence on mathematicians, nor on philosophers of mathematics" - see (7)-.
There is no mention of how Hobbes would have found (if he did have) this property of arc lengths of spiral and parabola and it appears it was a good intuition. Kirsti Pedersen in (4) writes : "Thomas Hobbes seems to be the first who got the idea to compare the arc lengths of the two curves at the end of 1642 and Roberval the first to give a proof. Mersenne informed Fermat and Torricelli of it. Both proved later that this property could be generalized to spirals $\rho^{n}=a \theta^{m}$ and parabolas $y^{m}=b \cdot x^{(m+n)}$ where m and n are rational numbers and $b=n^{m} /(n+m)^{m} \cdot a^{-1}$ ".

Later, under the pseudonim of Amos Dettonville, (10/12/1658) Pascal gave in (5) his own proof and writes :
"Thifteen years ago Mr Hobbes believed that the curve line of a given parabola was equal to a
right line. Mr de Roberval replied it was equal to the curve line of a given Archimede spiral; But without a demonstration other than motion, we can see some explanation in the book on Hydraulica (1644) of R.P. Mersenne : and since this kind of demonstration is not absolutely convincing, other geometers believed he was wrong, and published that this parabolic line was equal to the half circonference of a given circle." ..." And neither using the method of motions, nor the one of indivisibles, but following the the path of ancients so hencefor that could be strong and undisputed. I have done it and found that Mr Roberval was right and that the parabolic line and the spiral are equal one to the other".
It must be recalled that in 1642 , except the circle, only one plane curve, the logarithmic spiral, had been rectified by Descartes (1638) and independantly by Torricelli (1640) so the proof of equality of arc lengths between two plane curves was a surprising result. And it is only in 1657-58 that the semi-cubic parabola (Neil) and the cycloid (Wren) could be rectified.


Figure 3: Archimede spiral rolling on parabola

## 4 Geometria pars universali and Gregory's transformation

In 1668 James Gregory(1638-1675), a scottish geometer, was in Italy and published in 1668 at Padova a book "Geometria pars universalis" in which presented, in a geometric style, number of solutions to problems of the time. He also presents an interesting infinitesimal transformation which makes a correspondance between two plane curves in such a way it conserves the element of arc at corresponding points of the two curves, sometimes called "involuta" and "evoluta". This leads to a completely geometric result that we explain in the next section.
The Gregory's transformation (GT) : if the involuta is given in polar coordinates $(\rho, \theta)$ we get the evoluta parametric equation in an orthogonal frame $(y, x)$. If we respect some initial conditions, since it needs an integration, then the two curves of Gregory can roll without slipping on on the other in such a way that, if the evoluta is supposed to be fixed then the pole O of the evoluta will run on the x -axis. Conversely if we fix the involuta then the evoluta can roll on the involuta so the x -axis will constantly pass through the fixed pole O . This just like a couple wheel-ground ( = Involuta-Evoluta) many examples of such couples can be found as gif animations on Internet. Gregory' transformation gives the condition for the rolling curves and this is just the equality of arcs (no slipping) of the two curves.
But this way of seeing seems more recent and goes back to 19th century in 'Nouvelles Annales
of Mathematics'. The notations of Leibniz for calculus and polar coordinates help to present Gregory's transformation in a very simple set of equations.
Below we recall geometric properties of these corresponding couples of plane curves and use the terms ground $(y, x)$ and wheel $(\rho, \theta)$ to name the couple of associated objects defined by Gregory's transformation.

## 5 Polar coordinates, cartesian coordinates and the Gregory's transformation.

The rolling motion of a curve in a mobile plane on another curve in a fixed plane took part in $17^{\text {th }}$ century to the developpement of new methods on curves like the roulette or cycloid : the track of a point of a circle rolling on a straight line.
Modern notations developed by Leibniz give a clear exposition of the problem :


Figure 4: ground and wheel
We use the parametric equation $(y, x)$ or $(\rho, \theta)$ as functions of a single parameter to define the curves.
Gregory's transformation associates two plane curves, one in polar coordinates $(\rho, \theta)$ and the other in cartesian orthonormal $(y, x)$-frame, and is defined in the following way :
A - If the wheel is given we know $(\rho, \theta)$ then the ground has parametric equations:

$$
y=\rho \quad \text { and } \quad x=\int_{\theta o}^{\theta} \rho \cdot d \theta \quad \text { (1) Direct Gregory's Transformation: GT) } .
$$

B - In the opposite way, if the ground is given we know $(x, y)$ then the wheel has parametric equations:

$$
\left.\rho=y \quad \text { and } \theta=\int_{x o}^{x} \frac{d x}{y} \quad y \neq 0(2) \text { Inverse Gregory's Transformation : GT }{ }^{-1}\right) .
$$

$$
\text { Define : } \quad \tan V=\frac{\rho \cdot d \theta}{d \rho}=\frac{d x}{d y}(3)
$$

We give now the geometric properties of these corresponding ground $(x, y)$ and wheel $(\rho, \theta)$ to name the couple of curves associated by Gregory's transformation. It is a broad generalization of the the wheel, pole at the center.

Remark 1: For the wheel V is the usual angle between $\rho$ and the oriented tangent line. For the ground the angle V is defined between y'y and the oriented tangent. It is not the usual angle $\gamma$ between axe $\mathrm{xx}^{\prime}$ and the oriented tangent.
Remark 2: We note that $G T^{-1}$ is defined in the plane except on the line $y=0$. This line is called the "base-line" or "critical line".
Remark 3 : The rolling move is supposed to be without slipping so $s_{w}=s_{g}$. Since $\rho=y$ and $\rho d \theta=d x$ the two elements $d s$ arc lengths for two associated points are equal and :

$$
d s^{2}=d y^{2}+d x^{2}=d \rho^{2}+\rho^{2} d \theta^{2}=d \rho^{2} \cdot\left[1+\tan ^{2} V\right]=\left[\frac{d \rho}{\cos V}\right]^{2}=\left[\frac{\rho d \theta}{\sin V}\right]^{2} .
$$

The property 1 for areas is easy to verify :

$$
A_{\text {ground }}=\int y d x=2 \cdot \frac{1}{2} \int \rho^{2} d \theta=2 \cdot A_{w h e e l}
$$

The property 2 for equality for arc lengths is almost obvious :

$$
s_{\text {ground }}=\int \sqrt{d y^{2}+d x^{2}}=\int \sqrt{d \rho^{2}+(\rho . d \theta)^{2}}=s_{w h e e l}
$$

It is true for any couple of curves ground - wheel between two correponding bounds.

### 5.1 Triangle MTN in ground - Tangent subtangent / Normal subnormal (Leibniz differential triangle : dx, dy, ds)

$\mathrm{M}(\mathrm{x}(\mathrm{t}), \mathrm{y}(\mathrm{t}))$ is the current point of the curve then $\mathrm{n}=\mathrm{MN}$ is the normal, $\mathrm{t}=\mathrm{MT}$ the tangent, $\mathrm{sn}=\mathrm{HN}$ the sub-normal and $\mathrm{st}=\mathrm{HT}$ the sub-tangent. The ds is equal to the arc of circle passing through M centered at N and this variable circle has (C) as envelope. The differential triangle ( $\mathrm{dx}, \mathrm{dy}, \mathrm{ds}$ ) is similar to the THM (' is derivation wrt x ).

$$
\begin{aligned}
& y^{\prime}=\frac{d y}{d x}=\tan \gamma=\tan (\pi / 2-V) \quad d s^{2}=d x^{2}+d y^{2}=\frac{d x^{2}}{\cos ^{2} \gamma}=\frac{d y^{2}}{\tan ^{2} V} \\
& s t=-\frac{y}{y^{\prime}} \quad \text { sn }=y . y^{\prime} \quad t^{2}=y^{2}\left(1+\frac{1}{y^{\prime 2}}\right) \quad n^{2}=y^{2}\left(1+y^{\prime 2}\right)
\end{aligned}
$$

### 5.2 Polar Triangle MTN - Tangent subtangent / Normal subnormal (PolarTriangle : $d \rho, \rho d \theta$, ds)

$\mathrm{M}(\rho(t), \theta(t))$ is the current point of the curve then $\mathrm{pn}=\mathrm{MN}$ is the polar normal, $\mathrm{pt}=\mathrm{MT}$ the polar tangent, $\mathrm{psn}=\mathrm{ON}$ the polar sub-normal and $\mathrm{pst}=\mathrm{OT}$ the polar sub-tangent. The ds is equal for a couple of curves wheel-ground linked by Gregory' transformation (' is derivation wrt $\theta)$ :

$$
y=\rho \quad x=\int \rho d \theta \quad \text { or } \quad \rho=y \quad \theta=\int \frac{d x}{y} \quad y \neq 0
$$

For the wheel in polar coordinates, the differential triangle $(\rho . d \theta, d \rho, d s)$ is similar to the TOM.

$$
\frac{\rho}{\rho^{\prime}}=\frac{\rho d \theta}{d \rho}=\tan V=\tan (\pi / 2-\gamma) \quad d s^{2}=\rho^{2} d \theta^{2}+d \rho^{2}=\frac{d \rho^{2}}{\cos ^{2} V}=\frac{\rho^{2} d \theta^{2}}{\sin ^{2} V}
$$

$$
p s t=-\frac{\rho^{2}}{\rho^{\prime}} \quad p s n=\rho^{\prime} \quad p t^{2}=\rho^{2}\left(1+\frac{1}{\rho^{\prime 2}}\right) \quad p n^{2}=\rho^{2}\left(1+\rho^{\prime 2}\right)
$$



Orthonormal coordinates: $y=y(x) \Longleftrightarrow$ Ground

Figure 5: Triangle MTN in ground Orthonormal (y, x)


Figure 6: Triangle MTN in wheel $(\rho, \theta)$ the two triangles are superimposed at the contact point.

## 6 Couples wheel-ground : the geometric interpretation of Gregory's transformation

The transformation which associates a plane curve in polar coordinates $(\rho, \theta)$ to a plane curve in cartesian orthonormal coordinates $(x, y)$ is a mean to find the ground on which must roll the wheel with pole at O so that O runs along the x -axis of the fixed ground. And in the opposite way to find the wheel when the ground is given.

The Spiral of Archimede can roll inside a parabola in such a way that the pole of the spiral (the wheel) $\rho=a . \theta$ describes the axis (=the base-line) of the parabola $y^{2}=2 a x$ (the ground) Applying GT we have : $y=\rho=a . \theta$ and $x=\int \rho \cdot d \theta=\frac{a}{2} \theta^{2}$. so the ground is $y^{2}=2 . a \cdot x$. For the generalized archimedean spirals $\rho=\theta^{n}$ :

Applying GT we have : $y=\rho=a \cdot \theta^{n}$ and $x=\int \rho \cdot d \theta=\frac{a}{n+1} \theta^{(n+1)}$ so the ground is $y^{n+1}=$ a. $(n+1)^{n} \cdot x^{n}$.

For the generalized parabolas or hyperbolas of the form $y=a \cdot x^{n}$ with use of $G T^{-1}$ we find : $\rho=y=a \cdot x^{n}$ and $\theta=\int\left(a \cdot x^{n}\right)^{-1} \cdot d x=\frac{1}{a} \cdot \frac{x^{1-n}}{(1-n)}$ so the wheel is $\rho^{(1-n)}=a \cdot(1-n)^{n} \cdot \theta^{n}$.


Wheel

$$
\mathrm{MH}=\mathrm{y}=\mathrm{a}
$$



Archimede Spiral / Parabola
$H N=s n=p s n=a$


## Tractrix Spiral MT=t=pt=a



Logarithmic Spiral V= Cste
$\mathrm{MN}=$ Normale $=\mathbf{a}$ Al Tusi / Cardan

Figure 7: Example of couples Wheel / ground with constant element in triangle : y, n, st, sn, t and $V$

References:
(1) The Work of Archimede - T. Heath English translation (1897 Cambridge).
(2) B. Cavalieri - Geometria indivisibilibus ( 1635 Bologna).
(3) Shailesh A Shirali - Marin Mersenne, 15881648 Resonance Mars 2013.
(4) Kirsti Pedersen - Roberval's comparison of the arclength of a spiral and a parabola (1970).
(5) Amos Dettonville(B. Pascal) - Egalite des lignes spirale et parabolique (1658).
(6) Hardy Grant - Hobbes and mathematics. In T. Sorell, Cambridge companion to Hobbes(1996).
(7) Wolgang Breidert - Les mathematiques et la methode mathematique chez Hobbes. Revue Internationale de Philosophie, Vol. 33, No. 129,1979.

This article is the $22^{\text {th }}$ on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.

Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves. Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq- Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr

