CAUSTICS BY REFLECTION AND CURVES OF DIRECTION LOOKING FOR EXAMPLES - Part XXV -

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Abstract

The "curves of direction" go back to G. Salmon and E. Laguerre who gave their name to these curves which have a link with caustics by reflection and interesting properties. Curves of direction are caustics for parallele light rays with the condition to be algebraic and to have an algebraic expression of the element of arc. Leibniz, Bernoulli brothers and l'Hopital knew that this special type of caustics had a rational element of arc.

1 Caustics and Envelopes

In the last decade of seventeenth century the Bernouilli brothers explored new problems of the new calculus using Leibniz methods and notations. Differential geometry was beginning and Marquis de l'Hopital, in his textbook of 1696 written from lessons by Johann Bernoulli, gave a summary of the new technics of calculus known at the time. Tangents, osculator circle, envelope theory, involute, evolute are explained to the few readers. The book of the marquis was the principal document used by pro and anti new calculus and had a great influence in France to propagate this new Calculus method in the presentation of Leibniz and brothers Bernoulli.

Usual method to find envelopes in the plane of a family of curve depending on a parameter t is given by the elimination of this parameter and derivative w.r.t. t of the curve equations.

$$f(x, y, t) = 0 \qquad \text{and} \qquad \frac{df_t}{dt} = 0 \qquad \text{(a)}$$

We use a simpler set of equations to look for envelope of a line function of parameter t :

$$f(x, y, t) = u(t)x + v(t)y + c(t) = 0 \qquad \text{and} \qquad \frac{df_t}{dt} = 0 \qquad \text{(b)}$$

1.1 Envelope of a line angulary fixed to a variable circle rolling on x'x-axis.

We use equations (b) to find the envelope of a diameter (angulary fixed) of a variable circle rolling on x'x. The diameter (d) is in vertical position when the move begins. The variable radius is $R = f(t)$. The angle of rotation of (d) is θ, the abcisse of I is $x_I(t)$ then :

$$\theta = \int_0^t 1.dt = t \qquad x_I(t) = \int_0^t f(t).dt \qquad y_I(t) = f(t) \qquad \text{droite :} \quad y = x/\tan t + f(t)$$

The last equation is the one of the diameter (d) with parameter t. So the parametric equations of the envelope of (d) are given by :

$$\boxed{x = f'(t)\sin^2 t \qquad y = f'(t)\sin t\cos t + f(t)}$$

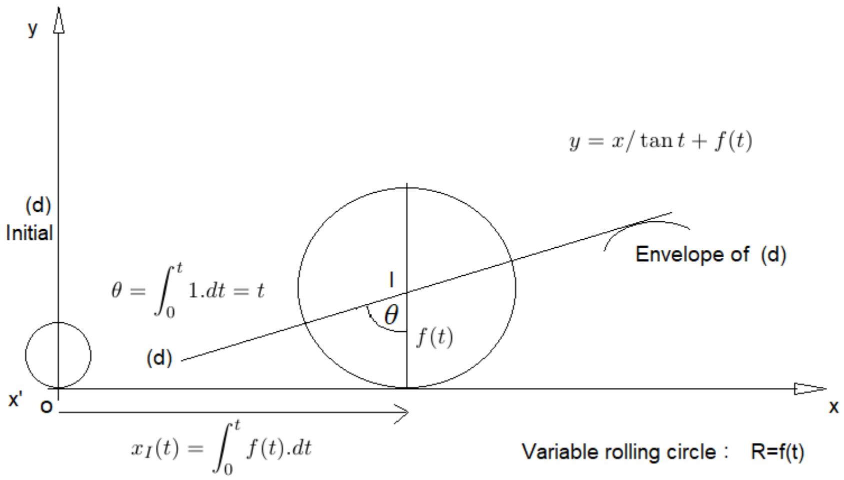

Figure 1: Variable Rolling Circle : envelope of an angulary fixed diameter

This method to consider a line moving in the plane by mean of a variable rolling circle is in fact equivalent to Euler magic equation :

$$x\sin\alpha - y\cos\alpha - p(\alpha) = 0 \qquad \text{or} \qquad y = x\tan\alpha - \frac{p(\alpha)}{\cos\alpha} = 0$$

This tangential representation of a curve by its tangents is mean to find properties of many curves when we choose the arbitrary function $p(\alpha)$.

1.2 Examples :

-1- If $f(t) = t$ then we obtain the cycloid :

$$x = \sin^2 t \qquad y = t + \sin t\cos t$$

-2- If $f(t) = \tan t$, we get :

$$x = \tan^2 t \qquad y = 2\tan t \qquad y^2 = 4.x \longleftarrow \text{A Parabola}$$

2 Curves of Direction

The concept was defined by G. Salmon in his treatise on plane curves and E. Laguerre who gave the name curves of direction (COD) in 1882. These authors limited the definition to algebraic curve (with algebraic element of arc function of the coordinates (x, y) expressed without radical). Appel and Humbert generalized to transcendental curves with examples as the classical catenary, singular curve of pursuit or the cycloid which have not a rational arc.
A definition of algebraic COD is the property : The distances of any point in the plane are rational function of the coordinates of the contact point in the plane. The tangential equation (ux+vy +1=0) is :

$$\boxed{f^2(u,v)(u^2+v^2) = F^2(u,v)}$$

This equation is the consequence of the formula giving the distance from a point M(x, y) to a line in the plane $MH^2 = (ux + vy + 1)^2/(u^2 + v^2) = f^2(u, v)$.
A curve $\Psi(x, y) = 0$ is of direction if $\Psi'^2_x(x, y) + \Psi'^2_x(x, y)$ is the square of a rational function of the coordinates of the points (x,y) of the curve.
Some properties :

- The inverse curve of a COD is a COD.
- The arc length s of a COD is expressed by the integral of a rational function of (x,y), coordinates of the current point.
- The caustics by reflection of algebraic curves are COD for parallele light rays.
- These caustics, in general, have an arc expressed by a rational function of the coordinates.
- The evolute of a COD is a COD except when the COD is cut orthogonally in 2 points by each normal.
- Any parallele curves of a COD is also a COD.
- The anticaustics of algebraic curves for parallele rays are COD.

Examples of COD are the nephroid, the Tschirnhausen's Cubic or the astroid. The first COD was probably the Nephroid of Huygens (1678) and E. W. Tschirnhaus published a paper on the caustic of the circle for parallele light rays in 1682 in Acta Eruditorum and the Tchirnhausen's cubic appears in the same publication in 1690.

3 Curves of Direction as caustics by reflection of algebraic plane curves for parallele light rays

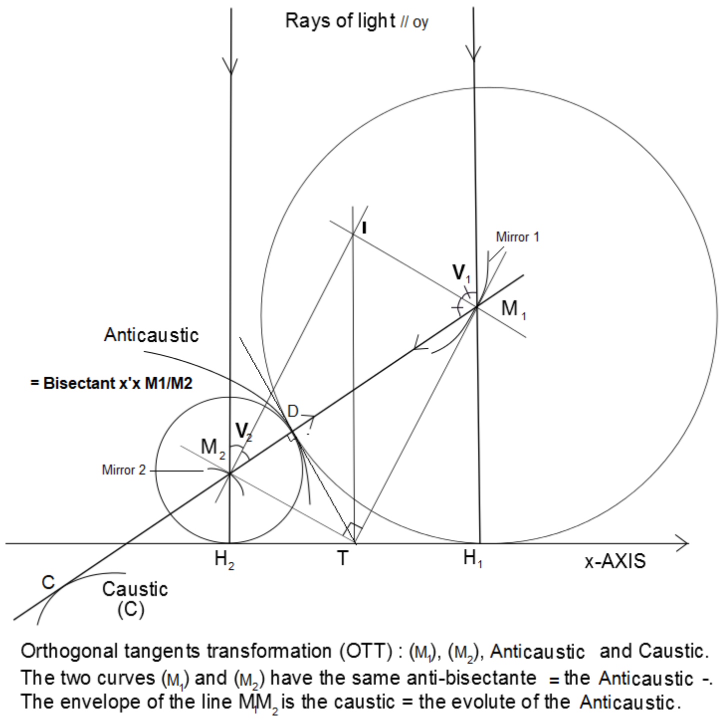

Figure 2: Two mirrors, anticaustic and its evolute the caustic

The caustic by reflection on a mirror curve in the plane is the envelope of reflected light rays coming parallele to Oy axis. It can be shown that there is another mirror on which the same ligth rays reflect and has the same caustic as envelope (see the figure above). These two mirrors M_1 and M_2 give the same caustic (C) and are exchanged by an orthogonal tangent transformation (OTT) with T moving on axis x'x. The projection D of point T on the reflected light ray is on the common anti-bisectant of the two mirror curves for the axis x'x. The caustic is envelope of the reflected light ray M_1M_2 and the evolute of the locus of D (anticaustic). The four curves : 2 mirrors /anti-bisectant and caustic form a quadruplet which will focus on in the rest of this paper.

The angle $(\overrightarrow{x'x}, \overrightarrow{TM_1}) = \gamma$ so that $\tan\gamma$ is the slope of TM_1 at current point M_1 on the

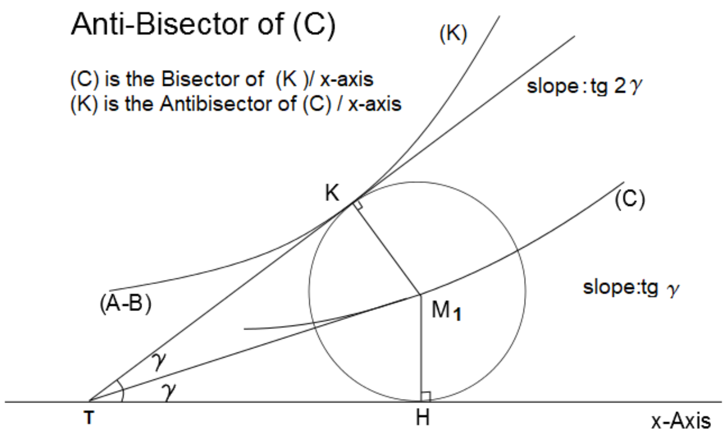

Figure 3: Anti-bisector

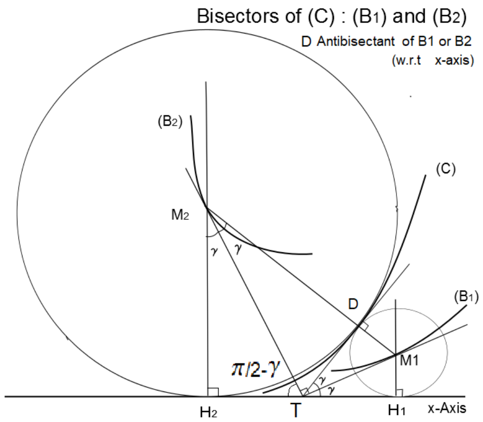

Figure 4: OTT and the two bisectors (B_1) and (B_2) of (C) w.r.t. x-axis

first mirror then for the second mirror curve at M_2, corresponding to M_1 by OTT, the slope is $\pi/2 - \gamma$. The common reflected ray on the mirrors is the line M_1M_2 : we can see on fig.4 that the slope of M_1M_2 is :

$$\boxed{\text{slope } M_1M_2 = 2\gamma - \pi/2}$$

So when the current tangent turns of du the line M_2M_1 turns of $-2.du$ and it is the property that explains the fact : the envelope of M_2M_1 is a caustic and if algebraic and under certain conditions it can be a COD. In this case the locus (C) of D is also a COD as involute of a COD (see fig.4).

For a given mirror and a direction of parallele light rays the caustic and the second mirror are unique. But in the other direction, given the caustic, there are infinitely possible pairs of mirrors to generate the caustic.

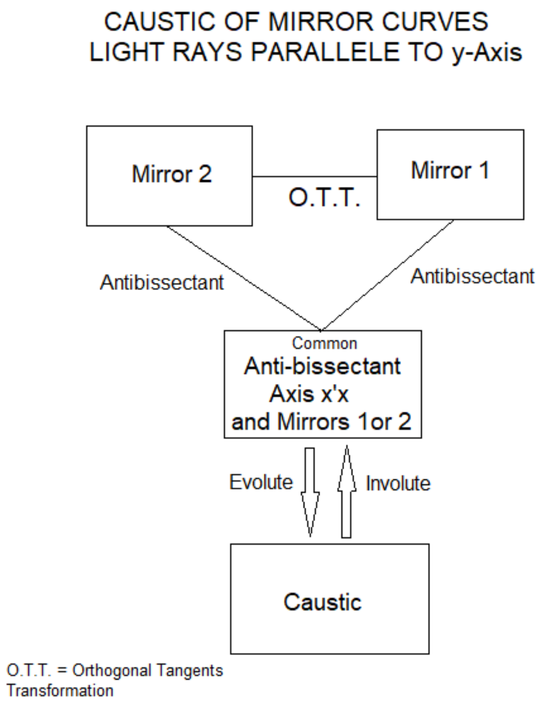

Figure 5: Links between the 4 curves : 2 mirrors 1-2, antibisectant and its evolute the caustic

4 Curves of direction obtained by the general formula with two arbitrary functions f(t), g(t):

The Tschirnhausen's cubic and the Nephroid are COD for which the mirrors were algebraic curves : parabola and circle. The rational expression of the arc length is a general property and if we look for curves in parametric coordinates x(t), y(t) we have the following general formulas depending on two arbitrary functions f(t) and g(t) then :

$$\boxed{x(t) = \int f(t).dt \qquad y(t) = \frac{1}{2}\int f(t)[g(t) - \frac{1}{g(t)}].dt}$$

The slope of the tangent at current point is :

$$\frac{dy}{dx} = \frac{g^2(t) - 1}{2g(t)}$$

so if we we choose for g a tangent function so $g(t) = \tan t$ then the slope is

$$-1/\tan 2t = \tan(2t - \pi/2) \qquad \text{so :} \quad \alpha = 2t - \pi/2$$

It is exactly the slope of M_1M_2 in the above geometric interpretation of caustics.
For these curves the arc length is given by an expression without radical :

$$\boxed{s_{to-t}(t) = \frac{1}{2}\int_{to}^{t} f(t).\left[g(t) + \frac{1}{g(t)}\right].dt}$$

So the expression under some assumptions can be algebraic and algebraically integrable. With these formulas we can generate as many CODs as we want. An easy example is the curve of pursuit of Bouguer. We set $f(t) = t^{n-1}$ and $g(t) = t$ then the parametric equations of the CODs depending on parameter n (ratio of speeds) are :

$$x(t) = \int t^{n-1}.dt \qquad y(t) = \frac{1}{2}\int t^{n-1}[t - \frac{1}{t}].dt$$

$$x(t) = \frac{t^n}{n} \qquad y(t) = \frac{1}{2}[\frac{t^{n+1}}{n+1} - \frac{t^{n-1}}{n-1}] \qquad s(t) = \frac{1}{2}[\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1}]$$

The function g(t) gives interesting properties if we choose for g(t) the tangent of an angle (see my paper 15). The Nephroid and the astroid are such examples with algebraic arc :

For the Nephroid : take $f(t) = 6\sin t^2 \cos t$ and $g(t) = \tan t$ then :

$$x = 2\sin t^3 \qquad y = (2\cos t^2 - 3)\cos t \qquad \longleftarrow \text{the Nephroid}$$

For the Astroid : take $f(t) = 3\sin t^2 \cos t$ and $g(t) = \tan t/2$ then :

$$x = \sin t^3 \qquad y = \cos t^3 \qquad \longleftarrow \text{the Astroid}$$

These general formulas can give curves which are transcendental COD with an arc not algebraic. Here are three such examples :
For the Cycloid take $f(t) = -4\sin t \cos t$ and $g(t) = \tan t/2$ then :

$$x = 2\cos t^2 \qquad y = 2t + \sin 2t \qquad \longleftarrow \text{the Cycloid}$$

For the special curve of pursuit take $f(t) = e^t$ and $g(t) = e^{-t}$ then :

$$x = e^t \qquad y = \frac{1}{4}(2t - e^{2t}) \qquad \longleftarrow \text{Special curve of pursuit}$$

For the Catenary take $f(t) = 1$ and $g(t) = e^t$ then :

$$x = t \qquad y = \cosh t \qquad \longleftarrow \text{the Catenary}$$

In this last case the mirrors are two exponentials $e^{\pm t}$ so not algebraic.

5 A method to find curves of direction from an known one :

In a paper of 1896 Paul Appell - see (5) - gives an interesting method to create new CODs when we know the equation a COD. We follow its paper. Suppose $F(X, Y) = 0$ the implicit equation of a COD. If S is the arc length of this curve then :

$$dS = \sqrt{dX^2 + dY^2} = R(X, Y)dX$$

where $R(X, Y)$ is rational function of X, Y. If $f(z)$ is a rational function of $z = x + iy$ (complex) choosed soo that all residues of $f^2(z)$ are equal to zero. We set :

$$Z = X + iY = \int f^2(z)dz \qquad (1)$$

The integral is a rational function of z. Identifying real and imaginary parts :

$$X = \phi(x, y) \qquad Y = \psi(x, y) \qquad (2)$$

By substituting these values in the equation of the initial COD $F(X, Y) = 0$ we get the equation of a new curve which is also a COD.
Equation (1) becomes :

$$dX + idY = f^2(z)(dx + idy)$$

Exchange now i in -i and use $\overline{z}$ the conjugate of z. We call $f(\overline{z})$ the function of conjugate variable. In general, if f is a holomorphic function whose restriction to the real numbers is real-valued or equivalently if the power serie representing $f(z) = \sum_0^\infty a_n z^n$ has only real coefficients a_n, and $f(z)$ and $f(\overline{z})$ are defined, then :

$$f(\overline{z}) = \overline{f(z)}$$

our function is supposed to have the property so the product $f(z).f(\overline{z}) = f(z).\overline{f(z)}$ is real so :

$$dX - idY = \overline{f^2(z)}(dx - idy)$$

$$dX^2 + dY^2 = f^2(z).\overline{f^2(z)}(dx^2 + dy^2)$$

and taking square root $dS = f(z).\overline{f(z)}ds$. Since the hypothesis $F(X,Y) = 0$ is a COD then dS/dX is a rational function of (X, Y) :

$$ds = \frac{R(X,Y)}{f(z).\overline{f(z)}}dX$$

So, from (2), ds is a rational function of (x, y) and the new curve $G(x,y) = 0$ is also a COD. The method requires to choose the function f(z) and a simple COD as starting curve to simplify the computations. The lower orders CODs are the line and the circle. There are only two CODs of third order : Tschirnhaus cubic and the special curve $x^3 - 3xy^2 = a^3$ or $r^3 \cos 3\theta = a^3$. Appell gives an infinite serie of COD using $f^2(z) = (2k+1)z^{2k}$ and for initial COD the line $X = a^k$ so we have :

$$X + iY = (2k+1)\int z^{2k}.dz = z^{2k+1}$$

And the new curves of direction are a subclass of sinusoidal spirals :

$$r^{2k+1}\cos(2k+1)\theta = a^{2k+1}$$

Taking $f(z) = 1/z$ reducts to the inversion in the plane.

Some more general sinusoidal spirals are also curves of direction :

$$\rho = \cos^{p/q}\left[\left(\frac{q}{p}\right).\theta\right] \text{ for odd p, q} \in \mathbb{N} \quad p \cap q = 1$$

6 Caustics by reflection and generalized sinusoidal spirals with $V = \pi/2 - 2t$

6.1 Sinusoidal spirals and Ribaucour curves

(We recall here some results of my paper 3)
The usual definition of Ribaucour curves is the following : given a fixed line (base line Δ) in the plane these are curves such that if MC is the radius of curvature the normal at the current point M cut the base line at N in a constant ratio k so : $\overline{MC}/\overline{MN} = k$. But we will give another equivalent definition. Ribaucour curves are grounds for sinusoidal spirals wheels when the pole runs on the base line Δ. The polar equation of sinusoidal spiral is $\rho = \cos(\theta/n)^n$ then by direct Gregory's transformation we have $y = \rho$ and $x = \int \rho d\theta = \int (\cos\theta/n)^n d\theta$. We set $u = \theta/n$ so $y = \cos^n u$ and $x = \int (\cos\theta/n)^n d\theta = n\int \cos^n u du$. These are the parametric equations of Ribaucour curves. We will generalize sinusoidal spirals. And the larger set of spirals will lead us to a larger set of curves that I call generalized Ribaucour curves as grounds corresponding to these new curves as wheels.

6.2 Some remarkable curves in polar parametric equations

Here are some polar parametric equations $(\rho(t), \theta(t))$ of well known plane curves as evolute of the circle, tratrix spiral, Norwich Spiral and its inverse :

$$\rho = 1/\cos t, \qquad \theta = \tan t - t \qquad \text{(Evolute of the circle)}$$

$$\rho = \cos t, \qquad \theta = \tan t - t \qquad \text{(Tratrix spiral)}$$

$$\rho = 1/\cos t^2, \qquad \theta = \tan t - 2t \qquad \text{(Norwich spiral)}$$

$$\rho = \cos t^2, \qquad \theta = \tan t - 2t \qquad \text{Wheel for the circle-ground and its tangent}$$

For these curves the angle V - between vector radius and tangent - $V = t$ for the first two and $V = \pi/2 - 2.t$ the two last. The polar angle is of the form : $\theta = n \tan t + p.t$ so these four curves have a linear expression of V function of t. We want to generalize and find other curves with analog property for V and θ expressed in the same form as above.

6.3 Two transformations keeping angle V

For the generalization we need two plane transformations :
The first one goes back to Mac Laurin and is the conformal complex function :

$$Z = z^n = \rho^n e^{in\theta} \qquad \text{with} \qquad z = \rho e^{i\theta}$$

which keep angle V and that we can write :

$$\rho \longrightarrow \rho^n \qquad \text{and} \qquad \theta \longrightarrow n.\theta \qquad \text{(Mac Laurin)}$$

The second transformation is the pedal, the locus of projection of pole O on the current tangent. Successive pedals have the same angle V. Initial curve is (ρ, θ) so :

$$\rho_{ped} = \rho.\sin V \qquad \text{and} \qquad \theta_{ped} = \theta - \pi/2 + V \qquad \text{(Pedal)}$$

Combining the two transformation (Mc Laurin/pedal) from the same pole also keeps angle V.

6.4 A special parametric expression for polar angle θ

Inspired by the four examples above we generalize the polar angle in the following form :

$$\theta = n \tan t + p.t$$

where n and p are real rational fixed parameter (or most often integers). Curves parametrized by angle t.

6.5 Grounds for generalised sinusoidal spiral for angle $V = \pi/2 - 2t$

We impose for our curves the angle V (between vector radius and tangent) to be equal to $\pi/2 - 2t$ so $\tan V = 1/\tan 2t$. Then from the formula:

$$\tan V = \rho d\theta/d\rho = \tan(\pi/2 - 2t)$$

we can compute by only one integration the values of ρ, since $\theta = n \tan t + p.t$ then

$$d\theta = [n(1 + \tan^2 t) + p]dt$$

And we get :

$$\frac{d\rho}{\rho} = [n(1 + \tan^2 t) + p] \tan 2t dt$$

$$\frac{d\rho}{\rho} = \frac{2n(1+\tan^2 t)\tan t}{1-\tan^2 t}dt + p\frac{\sin 2t}{\cos 2t}dt$$

$$\frac{d\rho}{\rho} = n\frac{2\tan t d(\tan t)}{1-\tan^2 t} + p\frac{\sin 2t}{\cos 2t}dt$$

and finally (by integration) :

$$\log(\rho/C) = -n\log|1-\tan^2 t| - \frac{p}{2}\log|\cos 2t| = -n\log|\frac{\cos 2t}{\cos^2 t}| - \frac{p}{2}\log|\cos 2t|$$

So the parametric equations of these curves are :

$$\rho\frac{(\cos t)^{2n}}{(\cos 2t)^{(2n+p)/2}} \qquad \theta = n\tan t + p.t \longrightarrow V = \pi/2 - 2t$$

We change first $p \longrightarrow 2p$ to keep integer indexes

$$\rho = \frac{(\cos t)^{2n}}{(\cos 2t)^{(n+p)}} \qquad \theta = n\tan t + 2p.t$$

and second $p \longrightarrow -(n+p)$ then the polar The equations of $C_{2*}(n,p)$ become :

$$\boxed{\rho = (\cos t)^{2n}.(\cos 2t)^{(p)} \qquad \text{and} \qquad \theta = n\tan t - 2(n+p).t}$$

This way p is exactly the pedal index since $V = \pi/2 - 2t$ and n the Mc Laurin index. In the expression of ρ the pedal and the Mc Laurin parts are separated.

7 A subclass of curves linked with wheels $C_{2*}(n,p)$ and to "curves of direction" as a generalization of Tschirnhausen's cubic or Nephroid:

We study grounds corresponding to wheels for which $p = 0$ with parametric equations :

$$\boxed{\rho = (\cos t)^{2n} \qquad \text{and} \qquad \theta = n(\tan t - 2t)}$$

We use these wheels in polar to find ground curves using direct Gregory's transformation ($y = \rho$ and x=$\int \rho.d\theta$) by one integration we get the parametric equation of the ground in the plane (x,y) for n positive integer :

$$y = \rho = \cos^{2n} t \quad \text{and} \quad x = \int \rho.d\theta = n\int \cos^{2n} t.(\tan^2 t - 1).dt$$

We identify these equations with the one of general caustics generated by the arbitrary functions f(t) and g(t) which are COD when algebraic. We find :

$$f(t) = 2n\cos^{2n-1} t\sin t \qquad \text{and} \qquad g(t) = \tan t$$

8 Ground corresponding to wheels $C_{2*}(n,0)$ when $n > 0$

The grounds corresponding to the class of wheels C_{2*}(n, p) are caustics by reflection in the same way as the well know example of Nephroid. The subclass of grounds corresponding to curves C_{2*}(n, 0) - with 2.n∈ Z - are caustics of plane curves.

we shall consider the curves for small n positive and negative. Since the ordinate $y = \cos^{2n} t$ we explore half integers (so : 2.n=an integer) and examine two classes $y < 1$ so n is positive and $y > 1$ for negative n. The first cases are :

n=1/2 : Poleni's curve,
n=1 : Circle,
n=3/2 : Nephroid,
n=-1 : Tschirnhausen's cubic,
n=-2 : l'Hopital quintic.

The following curves: $C_{*2}(n,0)$: $\rho = \cos^{2n} t, \quad \theta = n(\tan t - 2t)$ presents interesting

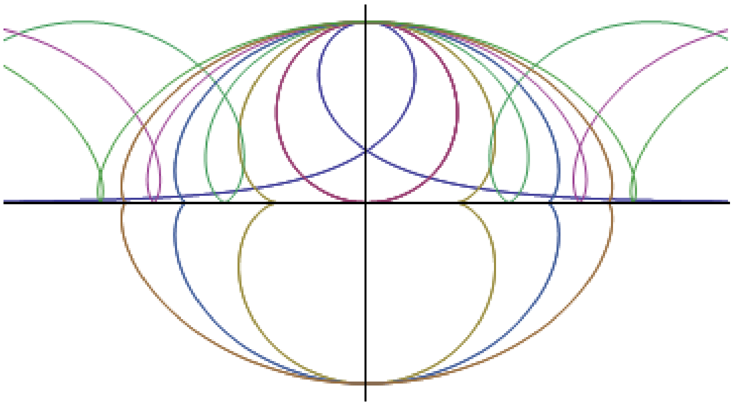

Figure 6: Ground for $C_{2*}(n,0)$ Angle $V = \pi/2 - 2u$ from n=1/2 to 4 by step 1/2
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properties : they are wheels for caustics. The serie of curves that we discuss here is a subserie of above grounds for the wheels $C_{2*}(n,p)$ when p=0 and n=1/2 to 4 step 1/2 :

n=	Curve	y=	x=
1/2	Poleni's curve	$1/\cosh t$	$(1/2)(t - 2\tanh t)$
1	Circle	$\cos^2 u$	$(1/2)\sin 2u$
3/2	Nephroid	$\cos^3 u$	$(1/2)\sin u(3 - 2\cos^2 u)$
2	—	$\cos^4 u$	$(1/2)[u + \sin 2u + (1/4)\sin 4u]$
5/2	—	$\cos^5 u$	$(1/2)\sin u[2.\cos^4 u + 2.\cos^2 u - 1]$
3	—	$\cos^6 u$	$(1/4)[(3u + 4\cos^3 u + 2\cos^2 u + 3\cos u)]$
7/2	—	$\cos^7 u$	$\frac{7}{2}[\frac{15}{32}\sin u + \frac{11}{96}\sin 3u + \frac{1}{32}\sin 5u + \frac{1}{224}\sin 7u]$
4	—	$\cos^8 u$	$4[\frac{15}{64}u + \frac{13}{64}\sin 2u + \frac{1}{16}\sin 4u + \frac{1}{64}\sin 6u] + \frac{1}{512}\sin 8u]$

As we noticed for positive values, in the table above for n=1 circle, 3/2 nephroid, 5/2 and 7/2 the curves are algebraic with a rational arc length and are curves of direction.

8.1 Curve of direction for n= 1 : Circle

We consider first the circle or more precisely two equal circles tangent externally placed symetrically w.r.t. Vertical axis (see fig.7) :

Curve	Mirror 1	Mirror 2
	$X_M = -1 + t\tan t/2$ $Y_M = \frac{t\cos t - \sin t}{-1+\cos t}$	$X_{M'} = 1 + t\tan t/2$ $Y_{M'} = \frac{-t\cos t + \sin t}{1+\cos t}$
$n = 1$	Anti-bisectant-Involute	Caustic = curve of direction
	$X_D = \cos t + t\sin t$ $Y_D = -t\cos t + \sin t$	$X_C = \cos t \pm 1$ $Y_C = \sin t$

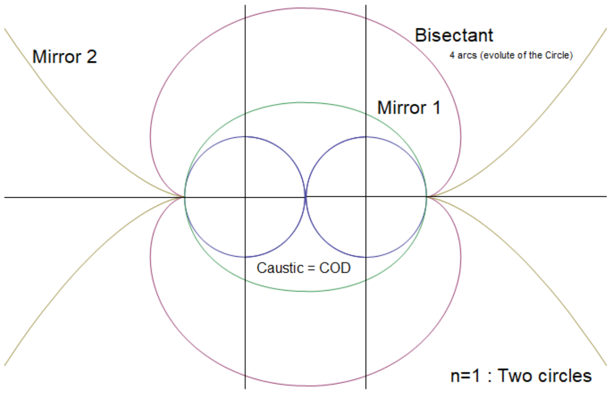

Figure 7: Quadruplet for n=1

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We have traced only a part of the first turn of the evolute of the circle : the two parts on either side of x-axis limited to the intersection with the y-axis and completed the graph by symmetry w.r.t. y-axis. We draw this way a simple curve made of two pieces thus avoiding drawing the involute (which is an infinite spiral) in its entirety. We find that the two mirrors and the involute are transcendental curves.

8.2 Curve of direction for n= 3/2 : Nephroid

The first curve is the nephroid (order 6) :

Curve	Mirror 1	Mirror 2
	$X_M = \cos t$ $Y_M = -\sin t$	$X_{M'} = 2/\cos t - \cos t$ $Y_{M'} = -\sin t\tan^2 t$
$n = 3/2$	Bisectant-Involute	Caustic = curve of direction
	$X_D = \cos^3 t + 3\cos t\sin^2 t$ $Y_D = 2\sin^3 t$	$X_C = \cos^3 t$ $Y_C = -(1/4)(3\sin t + \sin^3 t)$

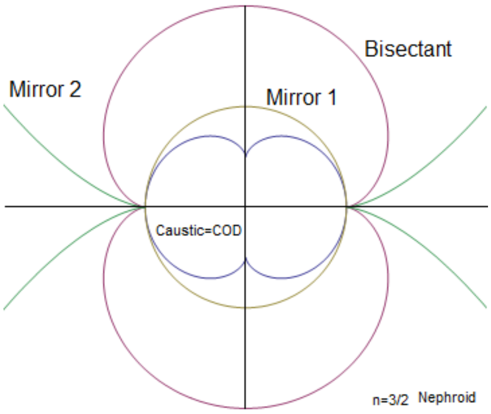

Figure 8: Quadruplet for n=1

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8.3 Curve of direction for n= 5/2.

Curve	Mirror 1	Mirror 2
	$X_M = (1/6)\cos t(5+\cos 2t)$ $Y_M = -(1/6)(9+\cos 2t)\sin t$	$X_{M'} = -(1/24)(-45+20\cos 2t+\cos 4t)/\cos t$ $Y_{M'} = -(1/6)(9+\cos 2t)\sin t\tan^2 t$
$n=1$	Anti-bisectant-Involute	Caustic = curve of direction
	$X_D = (1/24)(40\cos t - 15\cos 3t - \cos 5t)$ $Y_D = -(1/3)(9+\cos 2t)\sin^3 t$	$X_C = \cos^5 t$ $Y_C = (1/16)(-20\sin t - 5\sin 3t - 5\sin 5t)$

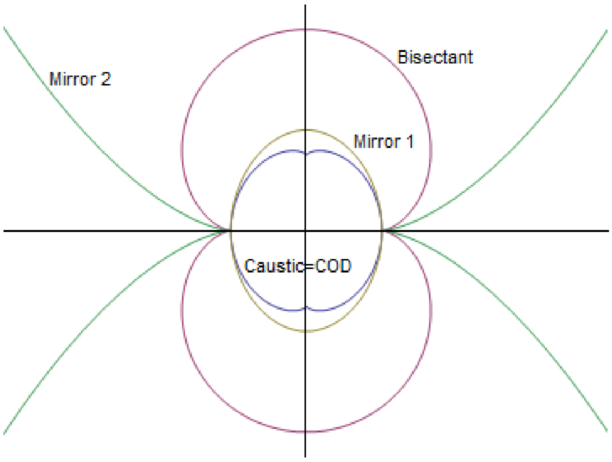

Figure 9: Quadruplet for n=5/2

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8.4 Curve of direction for n= 7/2.

Curve	Mirror 1
	$X_{M'} = (1/240)(206\cos t + 31\cos 3t + 3\cos 5t)$ $Y_{M'} = (1/120)(233 + 44\cos 2t + 3\cos 4t)\sin t$
	Mirror 2
	$X_M = (-1/160)(-350 + 175\cos 2t + 14\cos 4t + \cos 6t)/\cos t$ $Y_M = (1/120)(233 + 44\cos 2t + 3\cos 4t)\sin t\tan^2 t$
	Anti-bisectant-Involute
	$X_D = \cos^7 t + (7/120)\cos t(89 + 28\cos 2t + 3\cos 4t)\sin^2 t$ $Y_D = (1/60)(233 + 44\cos 2t + 3\cos 4t)\sin^3 t)\sin^3 t$
$n = 7/2$	Caustic = curve of direction
	$X_C = \cos^7 t$ $Y_C = (7/64)(15\sin t + (11/3)\sin 3t + \sin 5t + (1/7)\sin 7t)$

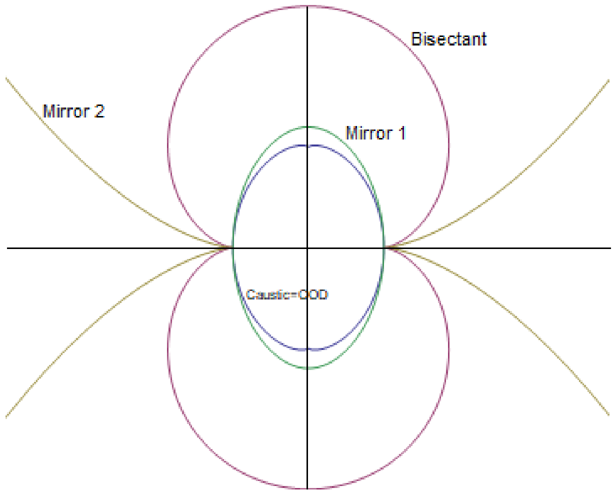

Figure 10: Quadruplet for n=7/2

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8.5 Curve of direction for n= 1/2. Poleni's curve

It is useful to consider instead of a unique curve, the union of two symmetrical copies of poleni's curves joined along the asymptote :

Curve	Mirror 1	Mirror 2
	$X_M = \cosh u$ $Y_M = u - \cosh u \sinh u$	$X_{M'} = u\cosh u$ $Y_{M'} = (1/2)(-1/\tanh u + u/\cosh^2 u)$
$n = 3/2$	Anti-bisectant-Involute	Caustic / not algebraic
	$X_D = (1 + u\tanh u)/\cosh u$ $Y_D = t/\cosh^2 u - \tanh u$	$X_C = 1\cosh u$ $Y_C = (1/2)(u - 2\tanh u)$

As we did above with the circle, for the Poleni's curve as caustic we have traced half of drawing on the right side of vertical axis and completed the graph by a symmetry w.r.t. its asymptote placed on the y-axis (see fig.11).

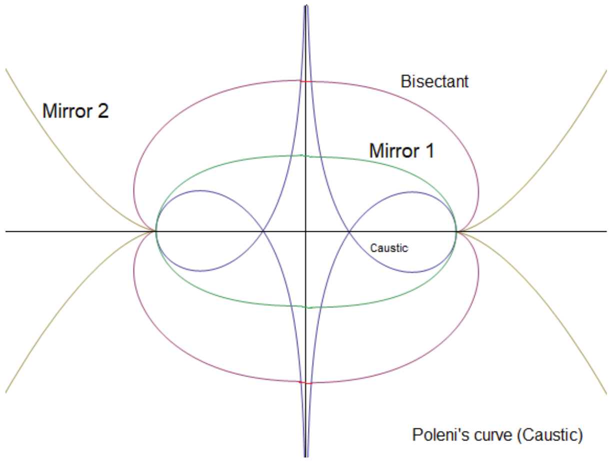

Figure 11: Quadruplet for n=1/2

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8.6 Ground corresponding to wheels $C_{2*}(n,0)$ when $n<0$

We calculate the first equations of the grounds for n=-1/2 to -4 with step -1/2. The polar angle θ is $\theta = n(\tan t - 2t)$ so $d\theta = n(\tan^2 t - 1)dt$. And we use a gudermanian change of variable $\cosh u = 1/\cos t$, $\sinh u = \tan t$ so $\cosh u.du = (1+\tan^2 t).dt = (1+\sinh^2 u).dt$ and $dt = du/\cosh u$ we get :

$$y = \cosh^{2n} u \quad \text{and} \quad x = \int \cosh^{2n} u.d\theta = n\int \cosh^{2n-1} u.(\sinh^2 u - 1).du$$

We identify these equations with the one of general COD generated by the arbitrary functions f(t) and g(t). We find easily :

$$f(t) = 2n\cosh^{2n-1} t \sinh t \qquad \text{and} \qquad g(t) = \sinh t$$

n=	Curve	y=	x=
$-1/2$	-	$\cosh t$	$(1/4)(\sinh t \cosh t - 3t)$
-1	T.C.	$1+v^2$	$v - v^3/3$ with : $v = \tan u = \sinh t$ (gudermanian)
$-3/2$	-	$\cosh^3 t$	$(3/8)[\tanh t \cosh^4 t - (5/2)[\tanh t \cosh^2 t - (5/2)t]$
-2	L.C.	$(1+v^2)^2$	$2[v^5/5 - v]$
$-5/2$	-	$\cosh^5 t$	$(5/48)\tanh t[4\cosh^6 t - 7\cosh^4 t - (21/2)\cosh^2 t - 15t]$
-3	-	$(1+v^2)^3$	$3[v^7/7 + v^5/5 - v^3/3 - v]$
$-7/2$	-	$\cosh^7 t$	$(5/48)\tanh t[4\cosh^6 t - 7\cosh^4 t - (21/2)\cosh^2 t - 15t]$
-4	-	$(1+v^2)^4$	$4(v^9/9 + 2v^7/7 - 2v^3/3 - v)$

In the table above the curves for n=-1, -2, -3 and -4 are algebraic with a rational arc length and are curves of direction stricto sensu. For n half integer the curves are not algebraic so not COD.

8.7 Curve of direction for n= -1 : Tschirnhausen's cubic.

This curve is a cubic :

Figure 12: Ground for $C_{-2}(n,0)$ Angle $V=\pi/2-2$ from n=-1 to - 4 by step -1/2

Curve	Mirror 1	Mirror 2
	$X_M=t^2$ $Y_M=2t$	$X_{M'}=-3t^2$ $Y_{M'}=2t^3$
$n=1$	Anti-bisectant-Involute	Caustic = curve of direction
	$X_D=t^2\frac{t^2-3}{1+t^2}$ $Y_D=\frac{4t^3}{1+t^2}$	$X_C=3t^2$ $Y_C=3t-t^3$

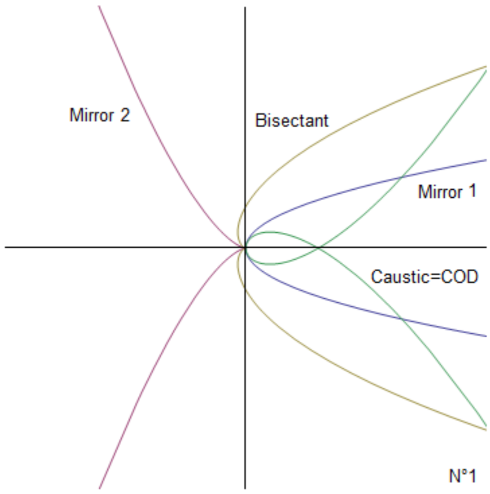

Figure 13: Quadruplet for n=-1

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8.8 Curve of direction for n= -2 : L'Hopital Quintic or Looping curve.

The second curve is of order 5 :

Curve	Mirror 1	Mirror 2
	$X_M = 1 + \frac{2t^2}{3} + \frac{t^4}{5}$ $Y_M = -\frac{4}{15}t(5 + t^2)$	$X_{M'} = 1 - \frac{t^2}{3}(6 + t^2)$ $Y_{M'} = \frac{4}{15}t^3(5 + t^2)$
$n = 2$	Anti-bisectant-Involute	Caustic $=$ curve of direction
	$X_D = \frac{(15-15t^2+5t^4+3t^6)}{(15+15t^2)}$ $Y_D = \frac{8}{15}t^3\frac{(5+t^2)}{(1+t^2)}$	$X_C = (1 + t^2)^2$ $X_C = \frac{2}{5}t((t^4 - 5))$

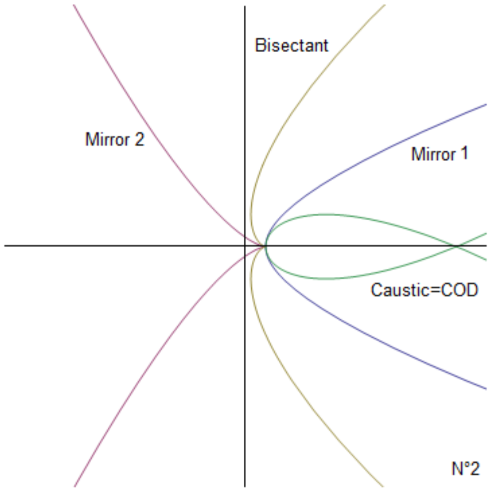

Figure 14: Quadruplet for n=-2

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8.9 Curve of direction for n= -3 :

This curve of direction is of order seven :

Curve	Mirror 1	Mirror 2
	$X_M = 1 + t^2 + \frac{3t^4}{5} + \frac{t^6}{7}$ $Y_M = -\frac{2}{35}t(35 + 14t^2 + 3t^4)$	$X_{M'} = 1 - \frac{t^2}{5}(t^4 + 5t^2 + 15)$ $Y_{M'} = -\frac{2}{35}t^3(35 + 14t^2 + 3t^4)$
$n = 3$	Anti-bisectante-Involute	Caustic $=$ curve of direction
	$X_D = \frac{35-70t^2+14t^6+5t^8}{35(1+t^2)}$ $Y_D = -\frac{4t^3(35+14t^2+3t^4)}{35(1+t^2)}$	$X_C = (1 + t^2)^3$ $Y_C = 3(\frac{t^7}{7} + \frac{t^5}{5} - \frac{t^3}{3} - t)$

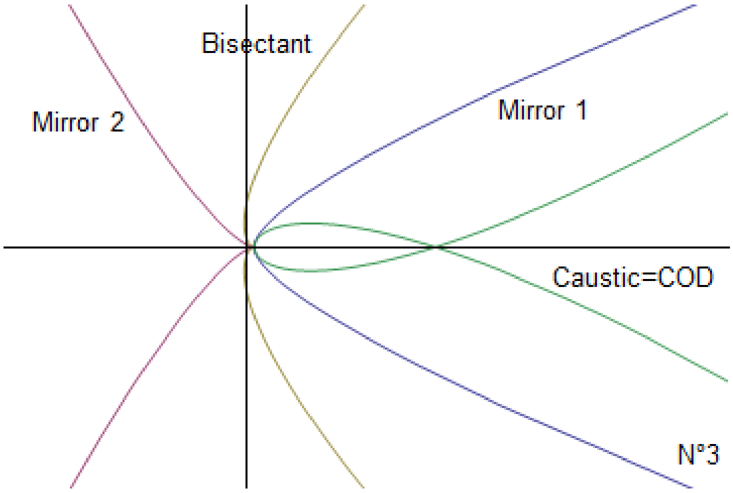

Figure 15: Quadruplet for n=-3

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8.10 Curve of direction n= -4

This curve of direction is of order 9 :

Curve	Mirror 1	Mirror 2
	$X_M = 1 + \frac{4t^2}{3} + \frac{6t^4}{5} + \frac{4t^6}{7} + \frac{t^8}{9}$ $Y_M = -\frac{8t}{315}(105 + 63t^2 + 27t^4 + 5t^6)$	$X_{M'} = 1 - 4t^2 - 2t^4 - \frac{4t^6}{5} - \frac{t^8}{7}$ $Y_{M'} = -\frac{8t^3}{315}(105 + 63t^2 + 27t^4 + 5t^6)$
$n = 4$	Bisectant-Involute	Caustic = curve of direction
	$X_D = \frac{315-945t^2-210t^4+126t^6+135t^8+35t^{10}}{315(1+t^2)}$ $Y_D = -\frac{16t^3}{315(1+t^2)}(105 + 63t^2 + 27t^4 + 5t^6)$	$X_C = (1 + t^2)^4$ $Y_C = 4(t^9/9 + 2t^7/7 - 2t^3/3 - t)$

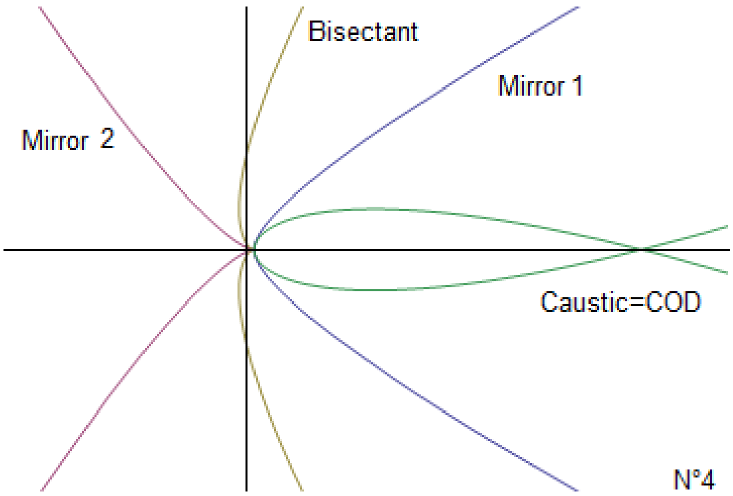

Figure 16: Quadruplet for n=-4

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9 Other classes of generalized sinusoidal spirals to explore.

The serie for p=0 is only a small part of curves $C_{2*}(n,p)$ but all values of p we can use the same procedure as the one explained above to find other plane caustics. And

also among them new curves of direction as ground corresponding to these generalised sinusoidal spirals.
An alternative possible choice, instead of $V = Pi/2 - 2t$ that led us to the curves of direction listed in present paper, is $V = 2t$ but this hypothesis for V gives the following parametric equation :

$$\rho = \frac{\sin^{n+p)/2} t}{\cos^{(n-p)/2} t}.e^{(-n/4)\tan^2 t} \qquad \theta = n \tan t + p.t$$

We find that the presence of a real exponential function $e^{(-n/4)\tan^2 t}$ significantly complicates the calculations. The three only cases without real exponential, as l have mentioned in my paper 3, are :

- 1 : $V = t$ for the class of curves $C_1(n, p)$,
- 2 : $V = \pi/2 - 2t$ for the class $C_{2*}(n, p)$,
- 3 : $V = 3t$ for the class $C_3(n, p)$.

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(2) Gomez-Teixeira - Traite des courbes speciales remarquables (1907)
(3) G. Salmon - Treatrise on the higher plane curves (1852)
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(5)P. Appell - Exercice sur les courbes de direction. N.A.M. 3e serie, tome 15 (1896), p. 491-495
A - Nouvelles annales de mathematiques (1842-1927) Archives Gallica
B - Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica

This article is the 25^{th} on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.

Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643 - 1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr