# A SELECTION OF SPECIAL PLANE CURVES $C_{k}(n, p)$ AND A FEW PROPERTIES CYCLODES <br> - Part XXVl - 

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10/04/2023


#### Abstract

We have selected some plane curves in the series of generalized sinusoidal spirals called $C_{k}(n, p)$. And give various properties using elementary geometry or analytic transformations like Gregory's and Mc Laurin's, pedals, evolutes and involutes. A section is devoted to Cyclodes of Sylvester.


## 1 Plane curves equations

Plane curves can be useful and sometimes have led to important developments.
So the Cycloid was a useful curve to test the first results (within reach of the geometers) of the calculus its specific properties was just at the level of what 17 century geometers could prove. This was the beginning of differential geometry.
The Lemniscate of Bernoulli was also an object of experimentation for Fagnano to find the first properties of elliptic functions before Euler, Gauss, Abel and Jacobi create the theory of elliptic functions.
Without forgetting the circle that lead to first the elementary transcendental functions.
We will now consider plane curves in polar and orthonormal coordinates given by parametric equations:

$$
\begin{equation*}
x=f(t) \quad \text { and } \quad y=g(t) \tag{a}
\end{equation*}
$$

or polar parametric equations :

$$
\begin{equation*}
\rho=f(t) \quad \text { and } \quad \theta=g(t)=0 \tag{b}
\end{equation*}
$$

and sometimes we use explicit equation when it is possible :

$$
y=f(x) \quad \text { and } \quad \rho=g(\theta) \quad(\mathrm{c})-(\mathrm{d})
$$

## 2 Curves with a special characteristic of angle $\gamma$ or $V$

When we compute areas between curves and line it is in general not too difficult to obtain by integration the result. But the computation of the arc length of a plane curve in orthonormal or polar coordinate system leads often to a difficult intgration. For the Lemniscate of Bernoulli, it needed to construct a new theory for the specific elliptic integrals. The reason is that most often the element of arc is expressed by a square root of a function and is not integrable.

$$
d s^{2}=d x^{2}+d y^{2} \quad \text { or in polar }: d s^{2}=(\rho d \theta)^{2}+d \rho^{2}
$$

### 2.1 Curves of direction

When this integration is possible (in general) the ds is rational, so the radical desappears, no square root in the expression. A classical example is illustrated by the curves of direction defined just by a constraint requiring the $d s^{2}$ to be a regular expression.

$$
x(t)=\int f(t) \cdot d t \quad y(t)=\frac{1}{2} \int f(t)\left[g(t)-\frac{1}{g(t)}\right] \cdot d t
$$

The slope of the tangent at current point is :

$$
\frac{d y}{d x}=\frac{g^{2}(t)-1}{2 g(t)}
$$

so if we we choose for g a tangent function so $g(t)=\tan t$ then the slope is

$$
-1 / \tan 2 t=\tan (2 t-\pi / 2) \quad \text { so }: \quad \alpha=2 t-\pi / 2
$$

For these curves the arc length is given by an expression without radical :

$$
s_{t o-t}(t)=\frac{1}{2} \int_{t o}^{t} f(t) \cdot\left[g(t)+\frac{1}{g(t)}\right] \cdot d t
$$

## 3 Some curves and their polar parametric equations

Here are some polar parametric equations $(\rho(t), \theta(t))$ of well known plane curves as involute of the circle, tratrix spiral, Norwich Spiral and its inverse :

$$
\begin{gathered}
\rho=1 / \cos t, \quad \theta=\tan t-t \quad \text { (Involute of the circle) } \\
\rho=\cos t, \quad \theta=\tan t-t \quad \text { (Tratrix spiral) } \\
\rho=1 / \cos t^{2}, \quad \theta=\tan t-2 t \quad \text { (Norwich spiral) } \\
\rho=\cos t^{2}, \quad \theta=\tan t-2 t \quad \text { (Central inverse of Norwich spiral) }
\end{gathered}
$$

For these curves we call V the angle between vector radius and tangent. This angle is a linear expression of parameter t . So $V=t$ for the first two and $V=\pi / 2-2 . t$ for the two last. The polar angle is in the form : $\theta=n \tan t+p . t$ and these four curves have a linear expression of $V$ function of $t$. We want to generalize this to find other curves with analog property for V and $\theta$ expressed in the same form as above.

## 4 Curves parametrized by a natural tangent.

Another way to find plane curves with a regular $d s$ is a class for which 1 call "parametrized by the tangent" of a parmeter $t$. For these curves we require that expression of the slope :

$$
\frac{d y}{d x}=\tan \gamma \quad \text { or in polar : } \quad \frac{\rho \cdot d \theta}{d \rho}=\tan V
$$

And we use $\gamma$ (orthonormal) or V (polar) - or a linear combination of this angle - as the parameter to express the coordinates. Since we have the classical formulas :

$$
d x=d s \cos \gamma \quad d y=d s \cdot \sin \gamma \quad \text { or in polar : } \quad \rho . d \theta=d s \sin V \quad d \rho=d s \cos V
$$



Figure 1: The circle with its first and second Involutes

Then we have a magic simplification, since :

$$
\begin{gathered}
d s^{2}=d x^{2}+d y^{2}=d s^{2} \cos \gamma^{2}+d s^{2} \sin \gamma^{2} \\
d s^{2}=(\rho . d \theta)^{2}+d \rho^{2}=d s^{2} \cos ^{2} \gamma^{2}+d s^{2} \sin ^{2} \gamma
\end{gathered}
$$

Then result is a square expression because we have chosen the tangent of a characteristic angle of the curve. We know some classical cases

### 4.1 Examples :

-1- If $x=-\sin t, y=\cos t$ then we obtain the circle with $\tan \gamma=\frac{\sin t}{\cos t}$ :

$$
d s^{2}=d x^{2}+d y^{2}=\cos ^{2} t \cdot d t+\sin ^{2} t \cdot d t=d t^{2}
$$

-2- If $x=t-\sin t, y=1-\cos t$, we get the cycloid and $\tan \gamma=\tan (t / 2)$ :

$$
d s^{2}=d x^{2}+d y^{2}=(1-\cos t)^{2} \cdot d t^{2}+\sin ^{2} t d t^{2}=2(1-\cos t) d t^{2}=4 \sin ^{2}(t / 2) d t^{2}
$$

These two curves are "naturally parametrized by the tangent" and so the trignometric expression of ds is free of its radical.

## 5 Generalized Sinusoidal spirals and Ribaucour curves $V=k . u$ or $V=\pi / 2-k . u$

All the curves we study have an angle $\theta=n \cdot \tan u+p . u$ we derived from constraint on V (to be equal to k.u ot $\pi / 2-k . u$ ) the radius vector $\rho$ of the curves in polar coordinates. In general we need a simple integration to get $\rho(t)$ (see Part lll). For the first values of k we get the following curves in polar parametric coordinates :
$k=1$ :

$$
\begin{array}{cl}
C_{1}(n, p): & \rho(u)=\tan ^{n} u \cdot \sin ^{p} u \quad \rightarrow V=u \\
C_{-1}(n, p): & \rho(u)=\frac{e^{\frac{n}{2} \tan ^{2} u}}{\cos ^{p} u} \quad \rightarrow V=\pi / 2-2 u
\end{array}
$$

$k=2:$

$$
C_{2}(n, p): \quad \rho(u)=\frac{(\sin u)^{(n+p) / 2}}{(\cos u)^{(n-p) / 2}} \cdot e^{-n / 4 \tan ^{2} u} \quad \rightarrow V=2 u
$$

$$
C_{-2}(n, p): \quad \rho(u)=\frac{(\cos u)^{2 n}}{(\cos 2 u)^{(n+p / 2)}} \quad \rightarrow V=\pi / 2-2 u
$$

$k=3:$

$$
\begin{gathered}
C_{3}(n, p): \quad \rho(u)=(\sin 3 u)^{(n+p) / 3} \cdot\left[\frac{3-\tan ^{2} u}{\cos u}\right]^{n} \quad \rightarrow V=3 u \\
C_{-3}(n, p): \quad \rho(u)=\frac{(\cos u)^{(4 n / 3)}}{(\cos 3 u)^{(4 n+3 p) / 9}} \cdot e^{(n / 6) \cdot \tan ^{2} u} \quad \rightarrow V=\pi / 2-3 u
\end{gathered}
$$

$k=4:$

$$
C_{4}(n, p): \quad \rho(u)=\frac{(\sin 4 u)^{(n+p) / 4} \cdot(\cos 2 u)^{n / 4}}{(\cos u)^{3 n / 2}} \cdot e^{(-n / 8) \cdot \tan ^{2}(u)} \quad \rightarrow V=4 u
$$

For $\mathrm{k}=4$ and $V=\pi / 2-4 u$, the expression $\rho$ is complicated with real exponentials.
We find that many of these equations present real exponentials that complicate the computation of the arc length. So we simplify the problem and give some selected examples without real exponential in the three special cases $\mathrm{k}=1,-2,3$.

## 6 The three special classes of generalized sinusoidal spirals :

In paper 1ll, 1 listed some classes of curves $C_{k}(n, p)$ in polar parametric coordinates depending on three integer parameters k (angle index), n (Mc Laurin index) and p (pedal index) with angle V equal to linear function of variable $\mathrm{u}: V=u, \quad V=\pi / 2-2 u$ and $V=$ $3 u$.

Class $C_{1}(n, p)$ :

|  | $\rho=\tan ^{n}(u) \cdot \sin ^{p}(u)$ | $\theta=n \tan (u)+p \cdot u \quad \rightarrow \quad V=u$ |
| ---: | :---: | :---: | :---: |
| or : | $\rho=\cos ^{n}(u) \cdot \sin ^{p}(u) \quad \theta=n \tan (u)-(n+p) \cdot u \quad \rightarrow \quad V=u$ |  |

Class $C_{2}(n, p)$ :
$\rho=(\cos u)^{2 n} \cdot(\cos 2 u)^{(p)} \quad \theta=n \tan u-2(n+p) \cdot u \rightarrow V=\pi / 2-2 u$
Class $C_{3}(n, p)$ :

$$
\rho=\left[\frac{\cos u}{3-\tan ^{2} u}\right]^{n} \cdot(\sin 3 u)^{p} \quad \theta=n \cdot \tan (u)-(n+3 p) \cdot u \quad \rightarrow V=3 u
$$

Many of the polar curves in this paper belong to one of this three classes ( $k, n, p$ ). Involute of circle $(1,1,-1)$ and Tractrix spiral $(1,-1,-1)$ to the first one $(k=1)$, Catalan curve ( -2 , $1,-1)$ and Norwich spiral $(-2,1,0)$ and its inverse $(-2,-1,0)$ to the second one $(\mathrm{k}=-2)$ and the and the evolute of the inverse of Norwich spiral belongs to the last class ( $\mathrm{k}=3$ ). In my notation this curve is $C_{3}(1,0)$

## 7 Curves Nr 1 : Evolute of Tractrix spiral or Catalan's curve.

The tractrix spiral is the central inverse of the involute of the circle and also the pedal of the Hyperbolic spiral. Its polar parametric equations are :

$$
\rho=\cos t \quad \theta=\tan t-t
$$



Figure 2: Tractrix spiral and its evolute (Catalan's curve)

$$
\begin{aligned}
d \rho & =-\sin t d t \quad d \theta=\tan ^{2} t d t \quad d s=\tan t d t \\
\tan V & =-\tan t \quad V=-t \quad R_{c}=\frac{\tan t}{\tan ^{2} t-1}=M C
\end{aligned}
$$

In the triangle $\mathrm{OMC}, \mathrm{M}$ current point and C the center of curvature, we have :

$$
\begin{gathered}
O C^{2}=O M^{2}+M C^{2}-2 \rho \cdot M C \cos \widehat{M} \quad O M=\rho=\cos t \\
O C^{2}=\cos ^{2} t+\left(\frac{\tan t}{\tan ^{2} t-1}\right)^{2}-2 \cos t\left(\frac{\tan t}{\tan ^{2} t-1}\right) \sin t \\
O C^{2}=\frac{1}{\left(\tan ^{2} t-1\right)^{2}} \cdot\left(\left(\tan ^{2} t-1\right)^{2} \cos ^{2} t+\tan ^{2} t-2 \cos t \tan t \sin t\left(\tan ^{2} t-1\right)\right) \\
O C^{2}=\frac{1}{\left(\tan ^{2} t-1\right)^{2}} \cdot\left(\left(\tan ^{2} t-1\right)\left(\sin ^{2} t-\cos ^{2} t-2 \sin ^{2} t\right)+\tan ^{2} t\right) \\
O C^{2}=\frac{1}{\left(\tan ^{2} t-1\right)^{2}} \cdot\left(\left(\tan ^{2} t-1\right)\left(-\sin ^{2} t-\cos ^{2} t\right)+\tan ^{2} t\right) \\
O C^{2}=\frac{1}{\left(\tan ^{2} t-1\right)^{2}} \cdot\left(\left(1-\tan ^{2} t\right)+\tan ^{2} t\right)=\frac{1}{\left(1-\tan ^{2} t\right)^{2}} \quad \text { so }: O C=\frac{1}{1-\tan ^{2} t}
\end{gathered}
$$

For the polar angle we use sinus-relations in the triangle OMC :

$$
\begin{gathered}
\frac{M C}{\sin \widehat{O}}=\frac{O C}{\sin \widehat{M}} \\
\frac{\tan t /\left(\tan ^{2} t-1\right)}{\sin \widehat{O}}=\frac{1 /\left(\tan ^{2} t-1\right)}{\sin \widehat{M}}
\end{gathered}
$$

Since $\widehat{M}=\pi / 2+t$ then $\sin \widehat{M}=\cos t$ and :

$$
\sin \widehat{O}=\cos t \cdot \tan t=\sin t
$$

So $\widehat{O}=t$ and the polar of point C is $\theta_{c}=(\tan t-t)+t=\tan t$. And the polar parametric equations of C so of the evolute of the tractrix spiral are :

$$
\rho_{c}=\frac{1}{1-\tan ^{2} t} \quad \theta_{c}=\tan t \quad \text { or } \quad \rho_{c}=\frac{1}{1-\theta_{c}^{2}}
$$

The evolute of the tractrix spiral is Catalan's curve.

## 8 Curves Nr 2 : Evolute of the wheel for the circle and its tangent.

In this section we search the polar parametric equations of the evolute of inverse of the Norwich spiral (Syvester 1868) $\rho(u)=1 / \cos ^{2} u \quad \theta=\tan u-2 u$. This curve is also the evolute of the wheel $C_{-2}(1,0)$ associated with the circle as ground and a tangent as base-line : $\rho(u)=\cos ^{2} u \quad \theta=\tan u-2 u$.


Figure 3: The wheel for the circle-ground and a tangent as base line.
This curve has interesting properties and an arc length equal to the one of the circle :

$$
\begin{gathered}
\rho=a \cos ^{2} t \quad \theta=2 t-\tan t \quad V=2 t-\pi / 2 \\
s=a . t \quad R_{c}=\frac{a}{\tan ^{2} t-3} \quad R_{c}=\frac{a}{\tan ^{2}(s / a)-3} \quad R_{c}=\frac{a \cdot \rho}{a-4 \rho}
\end{gathered}
$$

In the following we set $\mathrm{a}=1$ to simplify.

$$
d \rho=-2 \sin t \cos t d t \quad d \theta=\left(\tan ^{2} t-1\right) d t \quad d s=d t
$$

$$
\tan V=-\tan (\pi / 2-2 t) \quad V=2 t-P i / 2 \quad R_{c}=\frac{1}{3-\tan ^{2} t}=M C
$$

In the triangle OMC, M current point and C the center of curvature, we have :

$$
\begin{gathered}
O C^{2}=O M^{2}+M C^{2}-2 \rho \cdot M C \cos \widehat{M} \quad O M=\rho=\cos ^{2} t \quad \widehat{M}=2 t \\
O C^{2}=\cos ^{4} t+\frac{1}{\left(3-\tan ^{2} t\right)^{2}}-2 \cos ^{2} t \frac{1}{\left(3-\tan ^{2} t\right)} \cos 2 t \\
O C^{2}=\frac{1}{\left(3-\tan ^{2} t\right)^{2}}\left(\left(3-\tan ^{2} t\right)^{2} \cos ^{4} t-2\left(3-\tan ^{2} t\right) \cos ^{2} t \cos 2 t+1\right) \\
O C^{2}=\frac{1}{\left(3-\tan ^{2} t\right)^{2}}\left(16 \cos ^{4} t-8 \cos ^{2} t+1-16 \cos ^{4} t+12 \cos ^{2} t-2+1\right) \\
O C^{2}=\frac{4 \cos ^{2} t}{\left(3-\tan ^{2} t\right)^{2}} \quad \text { so }: \quad \rho_{c}=\frac{2 \cos t}{3-\tan ^{2} t}
\end{gathered}
$$

As above for the polar angle we need sinus-relations in the triangle OMC :

$$
\begin{gathered}
\frac{M C}{\sin \widehat{O}}=\frac{O C}{\sin \widehat{M}} \\
\frac{1 /\left(3-\tan ^{2} t\right)}{\sin \widehat{O}}=\frac{2 \cos t /\left(3-\tan ^{2} t\right)}{\sin \widehat{M}}
\end{gathered}
$$

Since $\widehat{M}=2 t$ then $\sin \widehat{M}=\sin 2 t$ and :

$$
\sin \widehat{O}=\frac{\sin 2 t}{2 \cos t}=\sin t \longrightarrow \widehat{O}=t
$$

And the polar angle of OC is :

$$
\theta_{c}=\theta_{M}+t=(\tan t-2 t)+t=\tan t-t
$$

Finally the polar parametric equations of the evolute of the curve are :

$$
\rho_{c}=\frac{2 \cos t}{3-\tan ^{2} t} \quad \theta_{c}=\tan t-t
$$



Figure 4: Inverse of Norwich spiral with its evolute

## 9 Curve Nr 3 : The wheel for a cardioid-ground and axis of symmetry as the base line

If the curve $C_{3}(1,0)$ rolls on a line the pole O describes a cardioid, the line is its axis of symmetry. By Steiner-Habich theorem (see part l) the pedal of this last curve is a wheel for a cardioid as the ground and the base-line is the axis of symmetry. For the parametric equations of the curve we just replace $p$ by $p+1$, so :

$$
C_{3}(1,1) \longrightarrow \rho(u)=a \cdot \cos ^{2} u \cdot \sin 2 u, \quad \theta=\tan u-4 u
$$

The arc length is $[2 a \sin u]_{0}^{u}$ (total length=4.a) and the radius of curvature is :

$$
R_{c}=\frac{d s}{d \theta+d V}=\frac{a \cos u}{\left(\tan ^{2} u-3\right) d u+(-3 d u)}=\frac{a \cos u}{\tan ^{2} u-6}
$$

So inflection points are in $\tan u= \pm \sqrt{6}$.
To find the ground we apply direct Gregory's transformation $\left(y=\rho, \quad x=\int \rho . d \theta\right)$ :

$$
y=a \cos ^{2} u \cdot \sin 2 u \quad x=a \int \cos ^{2} u \cdot \sin 2 u \cdot\left(\tan ^{2} u-3\right) d u=a \cos 2 u \cos ^{2} u
$$

These are equations of a cardioid with symmetry axis on x'x so our curve is a wheel for the cardioid (the base line is the axis of symmetry of the cardioid). The two curves have the same expression of arc length $s=2 a \sin u$, total length is 4 a . Its intrinsic equation is (a is homotety parameter) :

$$
R_{c}^{2} \cdot\left[7 s^{2}-6 a^{2}\right]^{2}=\left[a^{2}-s^{2}\right]^{3}
$$



Figure 5: Wheel rolling on a Cardioid base-line is the symmetry axis

## 10 The Cyclodes - J.J. Sylvester 1868

At the mathematic congress of 1868 in Norwich J.J. Syvester presented an interesting infinite family of plane cuves beginning with a circle. He remarked that the successive
involutes of a circle had some interesting properties that he exposed, later, in two papers of Philosophical Magazine.
The arc length is always accessible and there is an equation, he named "Arco-radial", which links the vector radius and the arc length.
The name of "Cyclodes" was suggested by A. Cayley. The first generation of the cyclodes are involutes of circle so all equals by rotation around the center of initial circle. The second generation, involutes of involute of circles present a large variety of forms among which we find some of this paper as Norwich Spiral or the second involute passing through the origin i.e. the center io initial circle. Cyclodes never have points of inflection, real or imaginary.
By definition the successive evolutes of any cyclode finish with the center of initial circle. We use the notations of Sylvester to give some useful formulas. $\phi, r, s, \theta$ denote the angle


Figure 6: Cyclodes of second generation -Transition case and Norwich spiral
position on initial circle of radius a, radius vector, arc length and vectorial angle of the curve. Starting with the circle : $s_{1}=a . \phi$ and for successives generations of involutes :

$$
s_{2}=a \frac{\phi^{2}}{2}+b \phi \quad s_{3}+a \frac{\phi^{3}}{6}+b \frac{\phi^{2}}{2}+c \phi \quad s_{i}=\int s_{i-1} d \phi
$$

We choose O as origin of coordinates at the center of initial circle. The perpendicular to the tangent is p and q the projection of the radius vector on the tangent. We have :

$$
\begin{array}{ccc}
q=-\frac{d p}{d \phi} & p^{2}+\frac{d^{2} p}{d \phi^{2}}=\frac{d s}{d \phi} & p^{2}+\left(\frac{d p}{d \phi}\right)^{2}=r^{2} \\
s^{\prime}=\frac{d s}{d \phi} & p^{\prime}=\frac{d p}{d \phi} & p^{\prime 2}=r^{2}-p^{2}
\end{array}
$$

There is relations for $r_{i+1}^{2}$ :

$$
\begin{gathered}
r_{i+1}^{2}=r_{i}^{2}-2 r_{i} s_{i} \frac{d r_{i}}{d s_{i}}+s_{i}^{2} \\
r_{i+1}^{2}=\left(s_{i}-s_{i-2}+s_{i-4 \ldots}\right)^{2}+\left(s_{i-1}-s_{i-3}+\ldots\right)^{2}
\end{gathered}
$$

For the ith involute to a circle, the arc is any integer rational function F of $\phi: \int F d \phi$ of degree $(\mathrm{i}+1)$ in $\phi$ and $r^{2}$ is a similar function of degree 2 i .

## 11 The second generation of cyclodes

The second generation is not too complicated :

$$
\begin{gathered}
r^{2}=\left(a / 2 \phi^{2}+b\right)^{2}+a \phi^{2} \\
\frac{d s}{d \phi}=a / 2 \phi^{2}+(a+b) \\
\theta=\arcsin \left(\frac{a \phi}{r}\right)+\phi
\end{gathered}
$$

There is a loop or cusps according as $\mathrm{a}+\mathrm{b}$ is positive or negative.

### 11.1 The transition case

Between this to possibilities we have the transition case when $\mathrm{b}=-\mathrm{a}$ :

$$
r^{2}=\frac{a^{2}}{4} \phi^{4}+a^{2} \quad s=\frac{a}{6} \phi^{2}
$$

So the arco-radial equation is : $\quad\left(r^{2}-a^{2}\right)^{3}=\frac{81}{4} a^{2} s^{4}$.
This involute is the locus of the centres of all circles cutting orthogonally the originating circle and the parent first involute since :
$p=a\left(\frac{\phi^{2}}{2}-1\right), \quad s^{\prime}=p+p^{\prime \prime}=a \frac{\phi^{2}}{2}, \quad r^{2}=p^{2}+p^{\prime 2}=a^{2} \frac{\phi^{4}}{4}+a^{2}, \quad$ and so $: \quad r^{2}-a^{2}=s^{\prime 2}$

### 11.2 Norwich spiral

This case corresponds to $b=-\frac{a}{2}$

$$
\begin{aligned}
r & =\frac{a}{2}\left(\phi^{2}+1\right) \quad s=\frac{a}{2}\left(\frac{\phi^{3}+3 \phi}{3}\right) \\
9 a s^{2} & =(2 r-a)(r+a)^{2} \leftarrow \text { Arco-radial equation }
\end{aligned}
$$

This curve is so that the radius vector OM is equal to the radius of curvature at each point. This is just the fact that using exprssion above :

$$
R_{c}=\frac{d s}{d \phi}=r
$$

### 11.3 Cyclode passing through origin

In this case the cyclode has the following polar parmetric equations :
Cyclode $2^{\text {nd }}$ generation through $\left.\mathrm{O}: \quad r=a \cdot \sin u / \cos ^{2} u\right)$ and $\theta=2 \tan u-u$
So this curve is $C_{1}(-2,1)$.


Figure 7: Parabola rolling on a fixed Spiral of Archimede

## 12 A parabola rolling on a fixed Spiral of Archimede

We know that the spiral of Archimede is a wheel for the parabola ground and the pole runs along the axis coinciding with x'x. If we fix the spiral so that the parabola rolls on it (in the same conditions) then the axis of the parabola passes constantly through the origin O on the spiral (duality for couples wheel/ground). We begin with the curve $C_{1}$ which generates the roulette.

The line MD is the tangent to the involute of circle (fixed) antipedal of the Spiral of Archimede. $\mathrm{OM}=$ p.t and $H M=p t^{2} / 2$ so the point $\mathrm{F}, \mathrm{S}$ and A (see fig. 6) describes respectively curves with following polar equations (same polar angle as the spiral shifted of $-\pi / 2$ ).

$$
\begin{gathered}
\mathrm{F}: \quad r=(p / 2)\left(\theta^{2}-1\right) \quad \rightarrow C_{-2}(-1,0) \\
\mathrm{S}: \quad r=(p / 2) \cdot \theta^{2} \quad \rightarrow C_{1}(-2,1) \\
\mathrm{A}: \quad r=(p / 2) \cdot\left(\theta^{2}+1\right)
\end{gathered}
$$

The locii of $\mathrm{C}, \mathrm{H}, \mathrm{D}$ are the respective antipedals of these three curves :

$$
\left.\mathrm{C}: \quad r=(p / 2) / \cos ^{2} u\right) \quad \text { and } \quad \theta=\tan u-2 u \quad \rightarrow C_{-2}(-1,0)
$$

$$
\left.\mathrm{H}: \quad r=(p / 2) \sin u / \cos ^{2} u\right) \quad \text { and } \quad \theta=2 \tan u-u \quad \rightarrow C_{1}(-2,1)
$$

$$
\mathrm{D}: \quad r^{2}=p^{2} / 4\left(1+6 \theta^{2}+\theta^{4}\right) \quad \text { and } \quad \theta=\theta_{M}+\arctan (2 / \theta)-\pi / 2 \quad \notin C_{k}(n, p)
$$

All points on the axis of the parabola run on a Galileo spiral : $r=(p / 2)\left(t^{2}-\lambda\right)$. With the parameter $\lambda=0$ at S so $r=(p / 2)\left(t^{2}\right)$. For the focus $\mathrm{F}: r=(p / 2)\left(t^{2}-1\right)$ and we know that the antipedal of this curve is the Norwich spiral.
The parallel to the axis of the parabola through M (instantaneous center of rotation) is
tangent to the antipedal of the spiral of Archimede which is the involute of the circle in the fixed plane. The motion generated by a parabola rolling on a fixed spiral of Archimede is equivalent to the rolling of a line on the involute of circle and so the locii of points on this line MC are involutes of involute of circle.
Norwich spiral, the locus of C , is one. The curve decribed by H, on tangent at the top of the parabola, in the fixed plane is an involute of an involute of circle passing through the origin O .
The curves described by F and C are in the class $C_{-2}(n, p)$ and the curves of S and H are in the class $C_{1}(n, p)$ also containing the spiral and the involute of circle in the fixed plane.

## 13 The ground rolling on the fixed wheel

In the last section we have seen an example of rolling ground on a fixed wheel. We generalize now this to any couple of wheel/ground associated by Gregory's transformation. We start here with the wheel and its pole O . The ground tangent at point M and the base line linked passes continually through the pole O. Another curve in the fixed plane will be useful : the antipedal of the wheel so the envelope of the perpendicular to the vector radius OM at M .


Wheel/Ground-Pedal/Antipedal

Figure 8: The general case of Steiner Habich theorem and curves correspondances
This figure, the rectangle of curves, which is an instantaneous symmetry picture of the movement evoluting with radius at current point $O M=\rho_{\text {wheel }}=y_{\text {ground }}$, as distance between the two parallel lines, can be read in two different ways (fixing ground or fixing
wheel) :
1 - The ground and base line are fixed : given a curve $\left(C_{1}\right)$ - and a pole M linked to it - rolling on a base line, the roulette of M is the ground for the pedal of $\left(C_{1}\right)$ w.r.t. M and placed in correct position of tangency by the symmetry w.r.t. the mediator $(\Delta)$ of the vector radius OM. This is Steiner Habich theorem.

2 - The wheel and its pole O are fixed : the ground rolls on the wheel, the base line passes through O constantly. The antipedal of the wheel is the envelope of the perpendicular to OM at M (in the fixed plane). The movement is equivalent to the one of a line rolling without slipping on the antipedal. All points on tangent line MP : $M_{1}, M_{1}, \ldots$ describe in the fixed plane an involute of this antipedal $\left(C_{1}^{\prime}\right)$. And the points on KO : $L_{1}, L_{1}, \ldots$ are pedals of $M_{i}$ locii. Note that the antipedal is nothing else than the curve $\left(C_{1}\right)$ transformed by symmetry w.r.t. the mediator $(\Delta)$ of radius vector OM. That symmetry places the curve $\left(C_{1}^{\prime}\right)$ in the position of the antipedal of the wheel. This can be considered as a kind of reciproque of the theorem of Steiner/Habich.

The two movements : wheel (rolling) on ground (fixed) and ground (rolling) on wheel (fixed) are direct/inverse corresponding to pedal/antipedal. The angle V is a common characteristic of all curves since the transformations, pedal, rolling and symmetry preserve this angle.

Let's summarize :
1 - Ground fixed : $\left(C_{1}\right)$ rolls on OK, pedal/M gives the wheel in tangent rolling position by pedal composed with symmetry $(\Delta)$.
2 - Wheel fixed : MP rolls on antipedal $\left(C^{\prime} 1\right)$ of the fixed wheel. $\left(C_{1}^{\prime}\right)$ is the symmetric of $\left(C_{1}\right)$ w.r.t. ( $\Delta$ ).
3 - Angle V between radius vector and tangent (up to sign) is common to all curves.
4 - Reversibility of the movements (ground or wheel fixed) is equivalent to performing a symmetry of axis ( $\Delta$ ).

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This article is the $26^{\text {th }}$ on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643-1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Part XXVl : A selection of special plane curves $C_{k}(n, p)$ and a few properties - Cyclodes
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr

