# SOME PLANE CURVES <br> - Part XXVII - 

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#### Abstract

Some plane curves are presented with various properties using elementary geometry, pedals, Mc Laurin and Gregory ( $G T / G T^{-1}$ ) transformations.


## 1 Antipedal and Euler normal equation of a line

This is a tangential representation of a curve (C) in orthonormal coordinates in the plane. This equation is the tangential equation of the antipedal of the curve $p=p(\theta)$ in polar coordinates.

$$
x \cos \theta+y \sin \theta-p(\theta)=0
$$

The distance OP orthogonal to the tangent from the the origin of coordinates is $p(\theta)$ in polar $(p, \theta)$. So the curve $(\mathrm{C})$ is the antipedal of this last curve. The parametric equations of the this antipedal are :

$$
x=p(\theta) \cos \theta-p^{\prime}(\theta) \sin \theta \quad y=p(\theta) \sin \theta+p^{\prime}(\theta) \cos \theta
$$

A classical example is the one of cycloidals considered as antipedals of rosaces $\rho=$ $\cos k \theta$ or $\rho=\sin k \theta$. Parametric equations are - here $p(\theta)=\cos k \theta)$, k is a rational parameter $\in Q$ - :

$$
x=\cos k \theta \cdot \cos \theta+k \sin k \theta \cdot \sin \theta \quad y=\cos k \theta \cdot \sin \theta-k \sin k \theta \cdot \cos \theta
$$

## 2 The antipedal of Clairaut' curves

These curves are defined in polar coordinates as $\rho=a \cos ^{n} \theta$ or $\rho=a \sin ^{n} \theta$. We set $\mathrm{a}=1$ w.l.o.g. The antipedals are the envelopes of the lines :

$$
x \cos \theta+y \sin \theta-\cos ^{n} \theta=0
$$

The envelope of this line depending of parameter $\theta$ is given by :

$$
x=\cos ^{n-1} \theta\left[n+(1-n) \cos ^{2} \theta\right] \quad \text { and } \quad y=(1-n) \cos ^{n} \theta \sin \theta
$$

### 2.1 The first cases of antipedals of $p(\theta)=\cos ^{n} \theta$.

With the above formulas, we list the cases $n \in[-3,+3]$ :

- $n=0$ : the circle $x=\cos \theta, \quad y=\sin \theta$.
- $n=1: \mathrm{x}=1, \mathrm{y}=0$ reduced to a point.
- $n=2: x=\cos \theta\left(1+\sin ^{2} \theta\right), y=-\cos ^{2} \theta \sin \theta$, an involute of the astroid passing through origin.
- $n=3: x=\cos ^{2} \theta\left(1+2 \sin ^{2} \theta\right), y=-2 \cos ^{3} \theta \sin \theta$, the deltoid.


Figure 1: Antipedals of $p(\theta)=\cos \theta^{n}$ for $\mathrm{n}=+2$ to +6

- $n=-1$ Parabola $x=\frac{\cos 2 \theta}{\cos ^{2} \theta}=1-\tan ^{2} \theta, \quad y=2 \tan \theta \quad x=1-(y / 2)^{2}$ the antipedal of the line : $p=\frac{1}{\cos \theta}$.
- $n=-2: x=\frac{1-3 \sin ^{2} \theta}{\cos ^{3} \theta}, y=3 \frac{\sin \theta}{\cos ^{2} \theta}$, antipedal of $p=\frac{1}{\cos ^{2} \theta}$.
- $n=-3: x=\frac{1-4 \sin ^{2} \theta}{\cos ^{4} \theta}, y=4 \frac{\sin \theta}{\cos ^{3} \theta}$ antipedal of $p=\frac{1}{\cos ^{3} \theta}$


Figure 2: Antipedals of $p(\theta)=\cos \theta^{n}$ for $\mathrm{n}=-1$ to -6

- $n=-p: x=\frac{1-(p+1) \sin ^{2} \theta}{\cos ^{p+1} \theta}, y=(p+1) \frac{\sin \theta}{\cos ^{p} \theta}$, antipedal of $p=\frac{1}{\cos ^{p} \theta}$.


### 2.2 The ground corresponding to Curves $C_{1}(n, 1)$

Curves $C_{1}(n, 1)$ are in the class $C_{1}(n, p)$, so have the following polar parametric equations:

$$
\begin{aligned}
& \rho=(1-n) \cos ^{n} u \sin u \\
& \theta=n \tan u-(n+1) u
\end{aligned}
$$

Where n and p , integers, are McLaurin and pedal index. Here $\mathrm{p}=1$ and by direct Gregory's transformation $\left(y=\rho, \quad x=\int \rho d \theta\right)$ we get:

$$
x=\int(1-n) \cos ^{n} u \sin u\left(n \tan ^{2} u-1\right) d u
$$

So the parametric equations of the ground are :

$$
\begin{gathered}
x=(1-n) \cos ^{n+1} u+n \cos ^{n-1} u \\
y=(1-n) \cos ^{n} u \sin u
\end{gathered}
$$

Up to a factor (1-n) the curve is the same as the antipedal of $p=\cos ^{n} \theta$. By application of Steiner Habich theorem the roulette of the pole of

$$
\rho=(1-n) \cos ^{n} u, \quad \theta=n \tan u-n u \quad \longleftarrow C_{1}(n, 0)
$$

is the curve ( $\mathrm{x}, \mathrm{y}$ ) above. Just replace $\mathrm{p}=0$ by $\mathrm{p}=1: C_{1}(n, 0)$ becomes $C_{1}(n, 1)$ the pedal which is a wheel for ground the roulette of $C_{1}(n, 0)$.

$$
\rho=(1-n) \cos ^{n} u \sin u, \quad \theta=n \tan u-(n+1) u \quad \longleftarrow C_{1}(n, 1)
$$

## 3 The antipedal of Galilean spirals $p(\theta)=\theta^{2}+h$.

Equation of these curves in polar coordinates is $p(\theta)=\theta^{2}+h$ where h is a constant parameter. The antipedals are the envelopes of the lines :

$$
x \cos \theta+y \sin \theta-\theta^{2}-h=0
$$

The envelope of this line depending of parameter $\theta$ is given by :

$$
\begin{equation*}
x=\left(\theta^{2}+h\right) \cos \theta-2 \theta \sin \theta \quad \text { and } \quad y=\left(\theta^{2}+h\right) \sin \theta+2 \theta \cos \theta \tag{1}
\end{equation*}
$$

In polar coordinate $(\rho, \phi)$ the parametric equations are :

$$
\begin{equation*}
\rho^{2}=\left(\theta^{2}+h\right)^{2}+4 \theta^{2} \text { and } \tan \phi=y / x=\frac{\left(\theta^{2}+h\right) \sin \theta+2 \theta \cos \theta}{\left(\theta^{2}+h\right) \cos \theta-2 \theta \sin \theta} \tag{2}
\end{equation*}
$$

These curves are the second involutes of the circle so are parallele curves in the plane. Two special cases correspond to $\mathrm{h}=0$ and $\mathrm{h}=-1$. If we set $t=\tan u$ then :
$\mathrm{h}=0: \quad \rho=4 \frac{\sin u}{\cos ^{2} u} \quad \theta=2 \tan u-u+\pi / 2 \longrightarrow C_{1}(-2,1) \quad$ Involute through origin and

$$
\mathrm{h}=-1: \quad \rho=\frac{1}{\cos ^{2} u} \quad \theta=\tan u-2 u \longrightarrow C_{-2}(1,0) \quad \text { Norwich Spiral }
$$

For this very special spiral, using polar formulas above (2), we have :

$$
\begin{gathered}
\rho^{2}=\left(\theta^{2}-1\right)^{2}+4 \theta^{2}=\left(\theta^{2}+1\right)^{2} \\
\tan \phi=y / x=\frac{\left(\theta^{2}-1\right) \sin \theta+2 \theta \cos \theta}{\left(\theta^{2}-1\right) \cos \theta-2 \theta \sin \theta}=\frac{\frac{\sin \theta}{\cos \theta}-\frac{2 \theta}{1-\theta^{2}}}{1+\frac{\sin \theta}{\cos \theta} \frac{2 \theta}{1-\theta^{2}}}
\end{gathered}
$$

If we set $\theta=\tan u$ then we get $\tan \phi=\tan (\theta-2 . u)$ so $\phi=\theta-2 . u=\tan u-2 . u$. And $\rho=1+\theta^{2}=\frac{1}{\cos ^{2} u}$.


Figure 3: Antipedals of $p(\theta)=\theta^{2}+h$ for $h=-1,0,+1$ and common evolute $(=$ involute of circle).

## 4 A quartic and its singular and ordinary focii in bipolar and tripolar coordinates.

The two following curves 1 and 2 are identical up to a dilatation :

### 4.1 Curve 1 defined by tripolar coordinates

$$
O M=\left|M F_{1}-M F_{1}^{\prime}\right|
$$

O is the midpoint of $F_{1} F_{1}^{\prime}=2 d$.

$$
\left(x^{2}+y^{2}\right)^{2}=\frac{4}{3} d^{2}\left(3 x^{2}-y^{2}\right)
$$

or :

$$
\rho^{2}=\frac{4 d^{2}}{3} \frac{\sin 3 \theta}{\sin \theta}
$$

4.2 Curve 2 defined by bipolar coordinates

$$
\frac{1}{M F_{2}^{2}}+\frac{1}{M F_{2}^{\prime 2}}=\frac{2}{d^{2}}
$$

Distance $F_{1} F_{1}^{\prime}=2 d$.

$$
\left(x^{2}+y^{2}\right)^{2}=d^{2}\left(3 x^{2}-y^{2}\right)
$$

or :

$$
\rho^{2}=d^{2} \frac{\sin 3 \theta}{\sin \theta}
$$

These curves have the same equation up to a dilation and are hyperbolic lemniscates with general equation :

$$
\left(x^{2}+y^{2}\right)^{2}=b^{2} x^{2}-a^{2} y^{2}
$$

Our two curves have special values of a and b with $a / b=\sqrt{3}$ :

Curve 1: $a=2 d$ and $b=\frac{2 d}{\sqrt{3}}$
Curve $2: a=d \sqrt{3}$ and $b=d$
On the x -axis hyperbolic lemniscates have four real focii :
1 - a pair of ordinary focii at $: \pm \frac{a b}{\sqrt{a^{2}+b^{2}}}$
2 - a pair of singular focii at $: \pm \frac{\sqrt{a^{2}+b^{2}}}{2}$
For curve 1 this gives : ordinary $\pm d$, singular $\pm 2 d / \sqrt{3}$
For curve 2 this gives : ordinary $\pm d \sqrt{3} / 2$, singular $\pm d$.
Since the distance between focii for the two curves is the same length 2 d ,

Then the curve 1 is defined w.r.t the ordinary focii, and the curve 2 is defined w.r.t the singular focii.


Figure 4: Tripolar and Bipolar definitions of the same plane quartic

$$
\begin{gathered}
\rho_{1}^{2}(\theta)=\frac{4 d^{2}}{3} \cdot \frac{\sin 3 \theta}{\sin \theta} \\
\rho_{2}^{2}(\theta)=d^{2} \cdot \frac{\sin 3 \theta}{\sin \theta} \quad \text { so }: \rho_{1}(\theta)=\rho_{2}(\theta) \cdot \frac{2}{\sqrt{3}}
\end{gathered}
$$

## 5 Caustic of the tractrix for light coming along y -axis

The tractrix is known since beginning of calculus, its parametric equations are :

$$
x=t-\tanh t \quad y=\frac{1}{\cosh t}
$$



Figure 5: Tractrix - anticaustic - catacaustic for parallel rays of light.

The anti-caustic is the locus of the symetrical of the foot of $y_{M}$ on x -axis w.r.t. the current tangent of the tractrix at M. We have $\tan \gamma=d y / d x=\left(\sinh t / \cosh ^{2} t\right) /\left(\tanh ^{2} t\right)=$ $1 / \sinh t$. And so $\cos \gamma=\tanh t$ and $\sin \gamma=1 / \cosh t$. This curve has two cups and can be computed by :

$$
\begin{gathered}
X=x-2 y \cdot \sin \gamma \cos \gamma \quad Y=2 y \cdot \cos ^{2} t \\
x=t+\tanh t-2 \tanh ^{3} t \quad y=\frac{2 \tanh ^{2} t}{\cosh t}
\end{gathered}
$$

The arc length of this anticaustic can be computed :

$$
d s=\left(2-3 \tanh ^{2} t\right) d t \quad \text { and } s=\int_{0}^{t} d s=[3 \tanh t-t]_{0}^{t}=3 \tanh t-t
$$

The radius of curvature is :

$$
\begin{gathered}
R_{c}=\frac{d s}{d \gamma}=\frac{\left(2-3 \tanh ^{2} t\right) d t}{d \gamma} \\
d \gamma=\frac{2 . d t}{\cosh t} \quad \text { so }: R_{c}=\frac{1}{2}\left(\frac{3}{\cosh t}-\cosh t\right)
\end{gathered}
$$

This helps to find the parametric equations of the evolute of the this last curve : it is the caustic of the tractrix for light rays parallel to $y$-axis.

$$
x=t-\tanh t+\tanh ^{3} t \quad y=\frac{1}{16}\left(\frac{15+\cosh 4 t}{\cosh ^{3} t}\right)
$$

## 6 Parallel curves to the catenary.

The equation of the plane catenary is

$$
x=t \quad y=\cosh t=\cosh x
$$

On the current normal of the catenary we set a point $\mathrm{MH}=\mathrm{h}$ a constant, so H moves on a parallele curve of the catenary. Elementary triangle : $\tan \gamma=\frac{d y}{d x}=\sinh t, \cos \gamma=1 / \cosh t$ and $\sin \gamma=\tanh t$. It follows easily that the locus of H is:

$$
x_{H}=t+h \cdot \tanh t \quad y_{H}=\cosh t-h / \cosh t \longrightarrow s_{H}=\tan v-v
$$



Figure 6: Parallele curves to the catenary and common evolute.

## 7 The syntractrix Catenary and Tractrix.

The Tractrix has, by definition, a constant tangent $=1$ and the syntractrix is the locus of a point fixed on the tangent to the tractrix so that $\mathrm{MT}=\mathrm{k} .1$, where M is on the syntractrix and $T$ on x'x. Poleni's curve is the case $\mathrm{k}=2$. The general parametric equation of syntractrices depending on k are :

$$
x(t)=t-k \tanh t \quad \text { and } y(t)=k / \cosh t
$$



Figure 7: Syntractrices.

## 8 Curves in form of a water drop - Cardioid - parabola

This curve is not unique and we can find many as propositions as we want for the form of a water drop curve. It must have a symetry axis, a round end and a cusp end. So I will


Figure 8: Three examples of drop curves
use two methods to generate water drop curves. The three above curves have repectively parametric equations :

$$
\begin{align*}
\rho(t)=5 / 3-\cos t \quad & \text { and }: \theta(t)=-t+4 \arctan (2 \tan t / 2)  \tag{A}\\
X(t)= & (\cos (t)+\sqrt{\cos 2 t}) \cos t \quad(B) \\
Y(t) & =(\cos (t)+\sqrt{\cos 2 t}) \sin t \\
X c & =\frac{3(2 \cos t+\cos 2 t)}{5+4 \cos t} \quad(C) \\
Y c & =\frac{3(2 \sin t+\sin 2 t)}{5+4 \cos t}
\end{align*}
$$

The first (A) is a wheel for the cycloid cusps upward (so has same length as an arch), the second (B) is an inverse of parabola and the third (C) is an inverse of a cardioid pole at the center of fixed circle.

### 8.1 Using a wheel for the cycloid

We have seen in paper V that a drop form can roll on the cycloid so that the pole run on x -axis. This is curve (A).


Figure 9: Drop curve as a wheel for the cycloid

### 8.2 Starting from a Parabola

It is possible by inversion to get a drop curve from a parabola. We set the center of inversion at the point of intersection of symetry axis and directrix. The result is in the fig. 6

### 8.3 Starting from a Cardioid.

The cardioid is not a drop, since the cusp is turned inside, but by inversion we may obtain a solution. This is easily done and the result is on the fig. 5.


Figure 10: Drop curves from parabola by inversion


Figure 11: Drop curve as inverse of a cardioid

## 9 A curve and its evolute



Figure 12: A curve and its evolute
If a tractrix spiral : $\rho=\cos t, \theta=\tan t-t$ rolls on a fixed line x 'x the roulette of the
pole is :

$$
x=\int d s-\rho \cos V \quad \text { and } y=\rho \sin V \quad \text { with } \quad d s=\tan t d t
$$

Here, since the tractrix spiral belongs to class $C_{1}(n, p)$ for $\mathrm{n}=1, \mathrm{p}=\mathrm{o}$, the angle $\mathrm{V}=\mathrm{t}$, the parameter.

$$
x=\cos ^{2} t-\log |\cos t| \quad \text { and } y=\cos t \sin t
$$

Now we look for a curve such that the segment cut on x -axis by the current tangent and normal is a constant length a. Subtangent is $S t=y / y^{\prime}$ and subnormal is $S n=y y^{\prime}$ so the geometric property is :

$$
S t+S n=a=y / y^{\prime}+y y^{\prime}=\left(y / y^{\prime}\right)\left(1+y^{\prime 2}\right)
$$

Since $y^{\prime}=\tan t$ this equation gives :

$$
y=\frac{a y^{\prime}}{1+y^{\prime 2}}=\frac{a \tan t}{1+\tan ^{2} t}=a \cos t \sin t
$$

And $y^{\prime}=\frac{d y}{d x}=\tan t$ and $d x=\int d y / \tan t$ solution of this equation is :

$$
x=\cos ^{2} t+\log |\sin t| \quad y=\cos t \sin t
$$

And the wheel w.r.t. x -axis is :


Figure 13: Wheels for the curve and its evolute


Figure 14: Wheels for the curve and for its evolute

$$
\rho=\cos t \sin t, \quad \theta=\tan t-2 t \quad \in C_{1}(n, p) \text { for } n=p=1
$$

This curve is the pedal of the tractrix spiral $(\rho=\cos t, \quad \theta=\tan t-t) \quad$ so $\mathrm{C} 1(1,0)$. By Steiner Habich theorem, this pedal is the wheel for the roulette of the pole of the tractrix spiral. The evolute of the curve : $x=\cos ^{2} t+\log |\sin t| \quad y=\cos t \sin t \quad$ is :

$$
x=\cos ^{2} t+\log |\sin t| \quad y=\cos ^{3} t / \sin t
$$

And the wheel for this last curve w.r.t. x -axis is :

$$
\rho_{1}=\cos ^{3} t / \sin t \quad \theta_{1}=3 \tan t-2 t \quad \in C_{1}(n, p) \text { for } n=3, p=-1
$$

We used to find this wheel the inverse of Gregory's Transformation $T G^{-1}$ :

$$
y=\rho, \quad x=\int \frac{d x}{y}
$$

The curve (roulette of tractrix spiral) is the first of a series with the following wheels :

$$
\rho=\cos ^{n} t \sin t \quad \theta=n \tan t-(n+1) t
$$

and the grounds given by direct Gregory's transformation :

$$
y=\rho \quad x=\int \rho d \theta
$$

The cases $\mathrm{n}=2$ corresponds to a ground involute of Astroid passing through origin and $\mathrm{n}=3$ to a Deltoid-ground. These are illustrated below.


Figure 15: A whell for the involute of astroid passing through the origin.


Figure 16: A wheel for the deltoid.

References:
(1) A - H. Brocard , T. Lemoine - Courbes geometriques remarquables Blanchard Paris (1967)
(2) Gomez-Teixeira - Traite des courbes speciales remarquables (1907)
(3)R. Deltheil - Les roulettes planes et l'integration e certaines equations differentielles - NAM (1927).

This article is the $27^{t h}$ on plane curves.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V: Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643-1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Part XXVI : A selection of special plane curves $C_{k}(n, p)$ and a few properties - Cyclodes
Part XXVII : Some plane curves.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr

