SOME SPECIAL ROULETTES AND ENVELOPES - Part XXVIII -

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Abstract

Some examples of general roulettes and envelopes of lines in the plane are presented specially by rolling curves as wheels adapted to a circular ground. We use classes of curves in $C_1(n,p)$, $C_{-2*}(n,p)$, $C_3(n,p)$ to exploit the simple lineary relation between θ and V that facilitates the calculations.

1 General equation of a roulette in polar coordinates

The general roulette in the plane is the locus of a point M fixed to a curve (C_1) rolling without slipping on a another curve (C),base in the fixed plane. A special case is when the curve (C) is a line in the fixed plane. It is convenient to use polar coordinates $\rho = \rho(\theta)$ with pole at M to simplify the calculations.

A crucial point is the possibility of a simple form for the length of the two curves and for angle V of (C) and (C_1) . Equal expression of arc length is rarely satisfied for arbitrarily selected curves. But for couples of curves associated by Gregory's transformation - so with same arc s - (or curves associated with the same curve by this transformation) it is possible. And curves of the class $C_k(n,p)$ are good candidates to get easy expressions for V and V_1. The base curve in

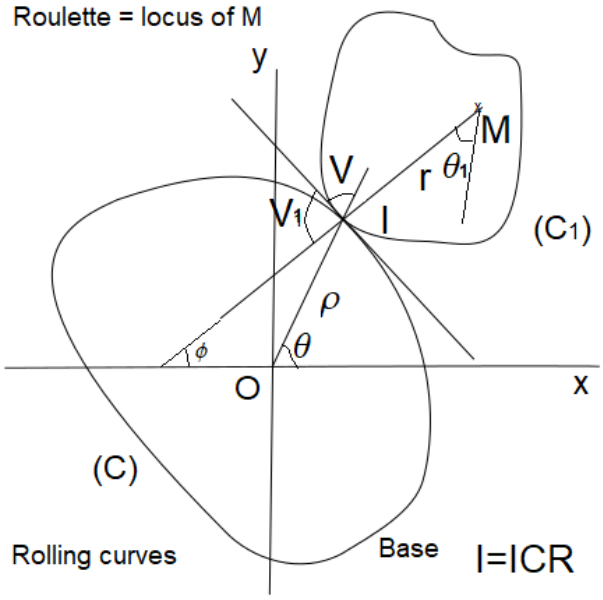

Figure 1: Roulette of a point M linked to a curve rolling on a base curve in the fixed plane

fixed plane with origin O of coordinates is $\rho(\theta)$ in polar (ρ, θ). The current tangent to the curve at I point of tangential contact with the rolling curve $r(\theta_1)$ in polar coordinates. The parametric equations of the locus the point M linked to the mobille plane are :

$$\overrightarrow{OM} = \overrightarrow{OI} + \overrightarrow{IM} = \overrightarrow{\rho} + \overrightarrow{r}$$

$$x = \rho(\theta)\cos\theta + r(\theta_1)\cos\phi \qquad\qquad y = \rho(\theta)\sin\theta + r(\theta_1)\sin\phi$$

$$\phi = \theta + V \pm V_1 \qquad \text{where } \phi = (\overrightarrow{Ox}, \overrightarrow{IM})$$

and V and V_1 are, as usual, the angles between vector radius and orientated tangent of base and rolling curve. The double sign $\pm$ depends on the inside or outside rolling (as for hypo-or epi-cycloidals).
For rolling curves without slipping the arc lengths are equal so : $s(u) = s_1(v)$. This constraint on integrals is the reason of the difficulty to give exact calculations in the general case.
The theory of roulettes was a subject of study in the 19th century but has lost some of its splendour but it is an interesting challenge because its a part of geometry.

2 Examples of applications to generalized sinusoidal spirals

The formulae in the above section can be used when curves are defined in polar coordinates with angle V depending of the same parameter u in linear form $V = ku$ or $V = \pi/2 - ku$. Where V is the angle between vector radius and oriented tangent. So these curves are generalized sinusoidal spirals : $C_k(n,p)$, k, n and p are respectively angle, Maclaurin and pedal index (k, n, p integers or eventually rational).
We have listed the first cases of these classes of curves (see part III) which have an essential property that it can be parametrized in polar coordinates by the angle u linearly linked with V so have a natural tangent parameter u. These curves have a rational element of arc ds (because of V= f(u), linear), sometime integrable. The most important constraint is the identity of the expressions of the arcs $s(u) = s_1(u)$ that must be verified systematically in the examples.

We know that the for some subclasses values corresponding to k=1 (V=u), k=-2 $(V = \pi/2 - 2.u)$ and k=3 (V=3.u) this three infinite classes give us interesting curves without real exponentials.

3 The curves chosen to illustrate the roulettes of generalized sinusoidal spirals

To begin we select a curve closely related to the circle : the wheel for a ground circle and a tangent as the base line. This curve has the following polar parametric equations :

$$\rho = 2\cos^2 u \qquad \theta = \tan u - 2u$$

It is the inverse w.r.t. the pole of Norwich spiral and belongs to the class $C_{-2}(n,p)$ for n=-1 an p= 0 so $C_{-2}(-1,0)$ and $\tan V = \pi/2 - 2u$ and arc length : s=2.u. Note that u is the angle in the center of the circle ground. V is a linear expression of u and that allows simpler calculations with above formulas.
The curve above that I call "spiral base" is the fixed base-curve on which a curve rolls and generates a roulette. In fact here we use only circles as rolling curves.

We know that the circle $\rho = 1 + \cos 2u = 2\cos^2 u$ with same arc length s=2.u and same parameter u.
We present two cases of rolling circles (C_1) on the base spiral for R=1 and R=1/2.

3.1 Rolling circle R=1

We know - by Gregory's transformation - that the circle with R=1 has same arc: $s_1(u) = 2u$ as the base spiral since this spiral is a wheel for this circle and angle in the center is double the inscribed angle. We look for a roulette of the point M on the circle so we need two forms of polar equation and the value of Angle V_1 :
For M in A at beginning of motion :

$$1 \rightarrow r = 2\sin u \qquad \theta = u \rightarrow \tan V_1 = \tan u \text{ so } V_1 = u$$

And for M opposit to A at beginning of motion :

$$2 \rightarrow r = 2\cos u \qquad \theta = u \rightarrow \tan V_1 = \tan(u - \pi/2) \text{ so } V_1 = u - \pi/2$$

3.2 Rolling circle R=1/2

The circle R=1/2, $r = (1/2)\sin u, \theta = u$ has an arc : $s_1(u) = u$ so if we modify the equation $\rho = (1/2)\sin 2u, \theta = 2u$ using same u as parameter and preserve the equality of rolling arcs, an essential constraint.
For M in A at beginning of motion :

$$3 \rightarrow r = (1/2)\sin 2u \qquad \theta = 2u \rightarrow \tan V_1 = \tan 2u \text{ so } V_1 = 2u$$

And for M opposit to A at beginning of motion :

$$4 \rightarrow r = (1/2)\cos 2u \qquad \theta = 2u \rightarrow \tan V_1 = -1/\tan 2u = \tan(2u - \pi/2) \text{ so } V_1 = 2u - \pi/2$$

Nota : This method may be generalized to circle R= p/q (rational) instead of 1/2 because of the property of angular at center and length of a circle arc (see fig. 7).

The point M is at a cusp when the circle rolls on an inflexion point of the base spiral which has two inflexion points corresponding to the $\tan u = \pm\sqrt{3}$. On some drawings (fig. 5) a position of the rolling circle is at an inflexion point (ICR) so M is at a cusp (Plucker relation). I use polar expressions (ρ_M, θ_M) for the roulette M.

3.3 A - Rolling circle R=1/2 (inside) on base Spiral and M opposit to A.

At the beginning of motion the fixed point M is diametrically opposed to A (fig. 2 bottom middle). The two curves are a couple of rolling curves around two poles so, if we fix one curve (the base), the pole of other runs around a circle centered in O. The application of formulae of section 1 gives :

$$\phi = \tan u - 2u + (\pi/2 - 2u) + (2u + \pi/2)$$

so

$$X_M = 2\cos^2 u \cos(2u - \tan u) - \cos 2u \cos(2u - \tan u)$$
$$Y_M = -2\cos^2 u \sin(2u - \tan u) + \cos 2u \sin(2u - \tan u)$$

which confirms the result : a circle centered at O with R=1.

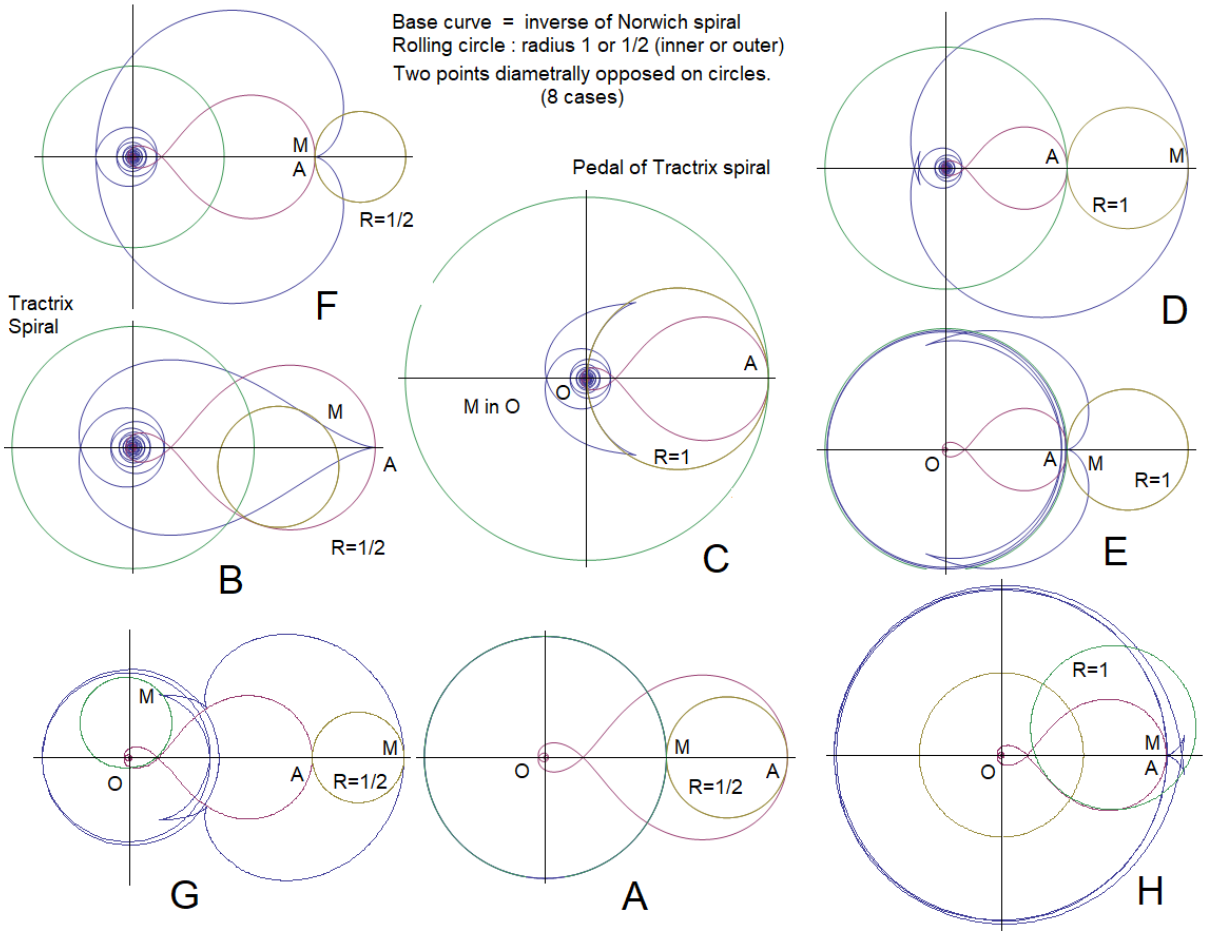

Figure 2: Roulette of a point on circle 1/2 or R=1 rolling inside/outside on base Spiral and M at A or opposit to A (2^3 cases). Base spiral : $\rho = \cos^2 u$, $\theta = \tan u - 2u$

3.4 B - Rolling circle R=1/2 inside on base Spiral and M at A.

At the beginning of motion the fixed point M is in A (fig. 3). The point L is on the circle (O, 1) and O, L and I the instant center of rotation (ICR) are aligned. The line MD is tangent to the roulette and orthogonal to IM. The angle $\widehat{MIL}$ is right since ML is a diameter. If C is at intersection circle (O, R=1) with MD : we have OC=OL=LM and CM is parallele to OL : OCML is a rhombus. So MC is constant. That's the geometric definition of the tractrix spiral. The formulae of section 1 give

$$\phi = \tan u - 2u + (\pi/2 - 2u) + 2u$$

so $\rho = 2\cos u$ and $\theta = \tan u - u$. Mathematica gives :

$$X_M = 2\cos^2 u \cos(2u - \tan u) + \sin 2u \sin(2u - \tan u)$$

$$Y_M = \cos(2u - \tan u)\sin 2u - 2\cos^2 u \sin(2u - \tan u)$$

but it may be reduced to the same equations.

3.5 C - Rolling circle R=1 (inside) on base Spiral and M at O opposit to A.

At beginning of motion M is opposit to A on the circle R=1. This case is the inverse motion of Gregory's transformation (couple ground/wheel). The wheel is fixed and the

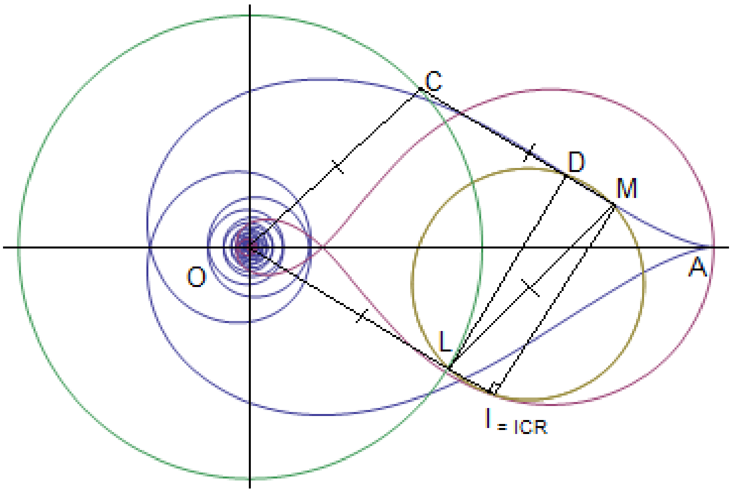

Figure 3: Tractrix spiral as roulette of a point on circle 1/2 on fixed base spiral.

circle rolls on the base spiral. Then the tangent to the circle at M passes constantly through the pole O (see fig.4).
In the rectangle OIMP the diagonal is $\cos t$ and angle $\widehat{OMP} = u$ so $PO = \cos u . \sin u$ and since OI is perpedicular to OP then $\theta = \tan u - 2u + \pi/2$. This geometric result is confirmed by the formulas in first section :

$$\phi = \tan u - 2u + (\pi/2 - 2u) - u - \pi/2$$

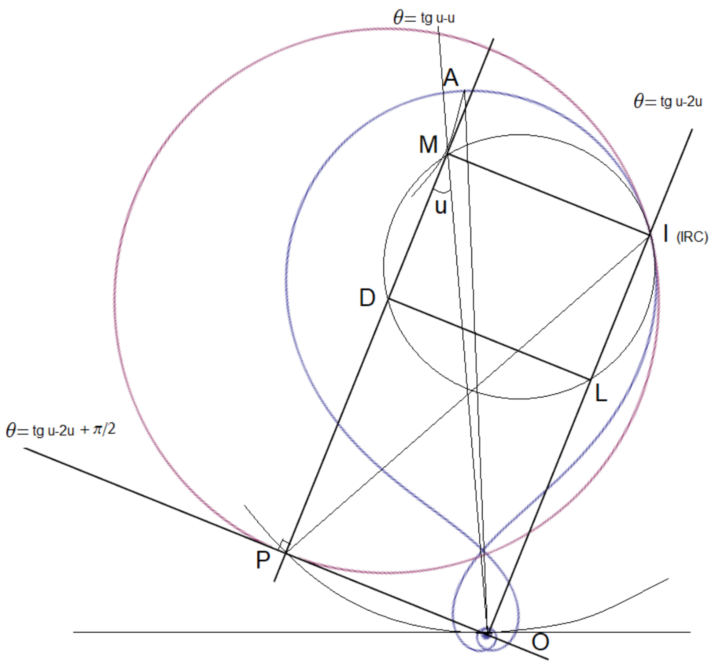

Figure 4: Pedal of Tractrix spiral as roulette of a point on circle 1 on fixed base spiral.

The locus of the pole or roulette is the following curve :

$$\rho = 2 \sin u \cos u \qquad \theta = \tan u - 2u + \pi/2 \qquad \longrightarrow C_1(1, -2)$$

This same curve is also the pedal of the Tractrix spiral with equations :

$$\rho = 2 \cos u \qquad \theta = \tan u - u \qquad \longrightarrow C_1(1, -1) \quad \text{with p (pedal index)} \rightarrow p - 1.$$

3.6 D - Rolling circle R=1 (outside) on base Spiral and M opposit to A.

Point M is opposit to A at the beginning of the motion. Formulae of section 1 give :

$$\tan u - 2u + (\pi/2 - 2u) - (u + \pi/2)$$

so the parametric equations are :

$$X_M = \cos^2 u \cos(2u - \tan u) + \cos u \sin(5u - \tan u)$$

$$Y_M = -\cos^2 u \sin(2u - \tan u) - \cos u \cos(5u - \tan u)$$

This curve seems to have a rational vector radius :

$$\rho_M^2 = \frac{1}{2}\cos^2 u(3 + 3\cos 2u + 2\cos 4u) \qquad 0 \le \rho \le 2$$

$$\tan\theta = \frac{Y_M}{X_M} = \frac{-\cos u \sin(2u - \tan u) - \cos(5u - \tan u)}{\cos u \cos(2u - \tan u) + \sin(5u - \tan u)}$$

I don't know if this expression of $\tan\theta$ can be reduce to a simpler formula, same remark for the other curves below.

3.7 E - Rolling circle R=1 (outside) on base Spiral and M in A.

Point M is in A at the beginning of the motion. We fix the first curve when the circle rolls without slipping. Here the formulae of section 1 are necessary and give :

$$\phi = \tan u - 2u + (\pi/2 - 2u) - u$$

The parametic equations of the roulette of M are :

$$X_M = \cos^2 u \cos(2u - \tan u) + \sin u \sin(5u - \tan u)$$

$$Y_M = \cos(5u - \tan u)\sin u - \cos^2 u \sin(2u - \tan u)$$

$$\rho_M^2 = \frac{1}{8}(9 + 2\cos 2u - \cos 4u - 2\cos 6u)$$

$$\tan\theta_M = \frac{\cos(5u - \tan u)\sin u - \cos^2 u \sin(2u - \tan u)}{\cos^2 u \cos(2u - \tan u) + \sin u \sin(5u - \tan u)}$$

The curve is drawn on fig. 2 and has a big loop, two cusps and two branches around the asymptotic point at origin O.

3.8 F - Rolling circle R=1/2 (outside) on base Spiral and M in A.

Point M is opposit to A at the beginning of the motion. Formulae of section 1 give :

$$\phi = \tan u - 2u + (\pi/2 - 2u) - 2u$$

so the parametric equations are :

$$X_M = 2\cos^2 u \cos(2u - \tan u) + \sin 2u \sin(6u - \tan u)$$

$$Y_M = \cos(6u - \tan u)\sin 2u - 2\cos^2 u \sin(2u - \tan u)$$

$$\rho_M^2 = 2\cos^2 u(2 + \cos 2u - \cos 6u)$$

$$\tan\theta_M = \frac{\cos(6u - \tan u)\sin u - \cos u \sin(2u - \tan u)}{\cos u \cos(2u - \tan u) + \sin u \sin(6u - \tan u)}$$

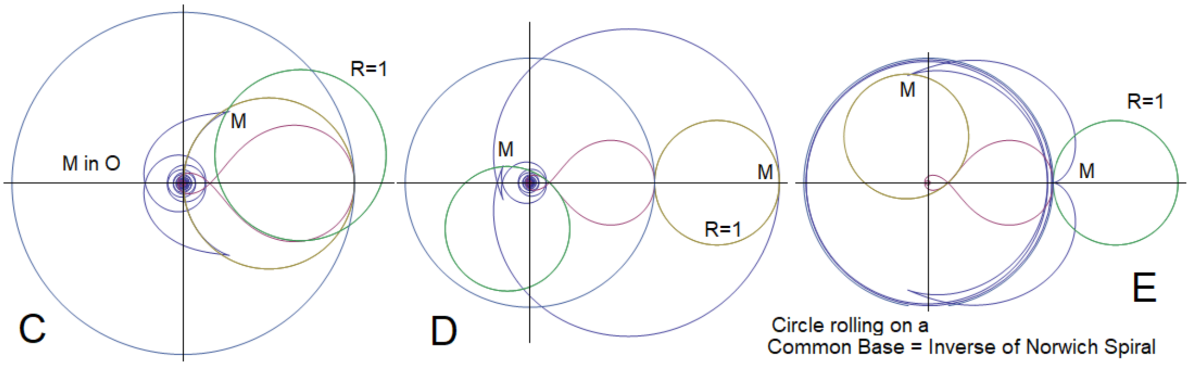

Figure 5: Roulettes point on a circle R=1 on base spiral and circle in position for the cusp.

3.9 G - Rolling circle R=1/2 (outside) on base Spiral and M opposit to A.

Point M is opposit to A at the beginning of the motion. Formulae of section 1 give :

$$\phi = \tan u - 2u + (\pi/2 - 2u) - (\pi/2 + 2u)$$

so the parametric equations are :

$$X_M = 2\cos^2 u \cos(2u - \tan u) + \cos 2u \cos(6u - \tan u)$$

$$Y_M = -2\cos^2 u \sin(2u - \tan u) - \cos 2u \sin(6u - \tan u)$$

$$\rho_M^2 = \frac{1}{2}(5 + 6\cos 2u + 4\cos 4u + 2\cos 6u + \cos 8u)$$

$$\tan\theta_M = \frac{-2\cos^2 u \sin(2u - \tan u) - \cos 2u \sin(6u - \tan u)}{2\cos^2 u \cos(2u - \tan u) + \cos 2u \cos(6u - \tan u)}$$

3.10 H - Rolling circle R=1 (inside) on base Spiral and M in A.

Point M is opposit to A at the beginning of the motion. Formulae of section 1 give :

$$\phi = \tan u - 2u + (\pi/2 - 2u) + u$$

so the parametric equations are :

$$X_M = 2\cos^2 u \cos(2u - \tan u) + 2\sin u \sin(3u - \tan u)$$

$$Y_M = 2\cos(3u - \tan u)\sin u - 2\cos^2 u \sin(2u - \tan u)$$

$$\rho_M^2 = \frac{1}{2}(9 - \cos 4u) \qquad 2 \le \rho \le \sqrt{5}$$

$$\tan\theta_M = \frac{2\cos(3u - \tan u)\sin u - 2\cos^2 u \sin(2u - \tan u)}{2\cos^2 u \cos(2u - \tan u) + 2\sin u \sin(3u - \tan u)}$$

$$\tan\theta_M = \frac{-4\sin(2u - \tan u) + \sin(4u - \tan u) + \sin(\tan u)}{4\cos(2u - \tan u) - \cos(4u - \tan u) + \cos(\tan u)} \qquad \text{given by Mathematica.}$$

4 Roulettes of circles on the first curve of Euler

Another example that can be illustrated using formulae of section 1 is to use as the base the curve found by Euler which has same length as the circle (3). Its polar parametric equations (given in part II) are :

$$\rho = \frac{2}{\sqrt{3}} + \cos t \qquad \theta = t - 4.\arctan\left[(2-\sqrt{3}).\tan\frac{t}{2}\right]$$

With the same procedure as for inverse of Norwich spiral in previous section, but we don't list the cases here, we obtain the eight curves in fig. 6 with shapes similar to those in fig. 2 of section 3.

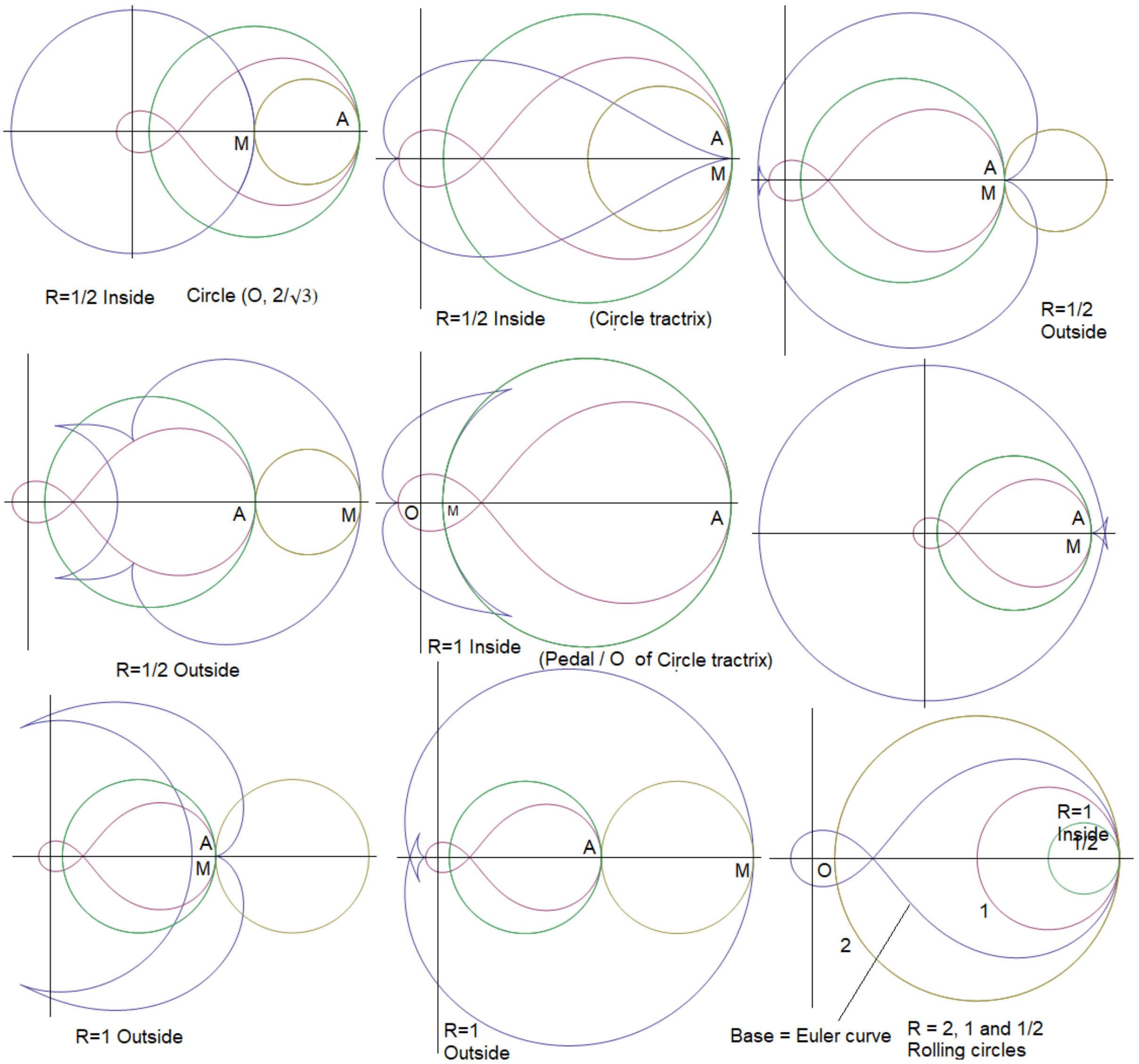

Figure 6: Roulette on Euler curve - rolling curves are circles R=1/2 or R=1.

4.1 Rolling circles on Euler curve

Using formulae of section 1 it is possible to draw the curves for dilated or reduced circles in commensurable proportion k with the circle in the definition of Euler curves : with equal arc as this circle. By analogy with classical cycloidals fig. 7 show some simple examples for 1/4, 1/6 and 3/2.

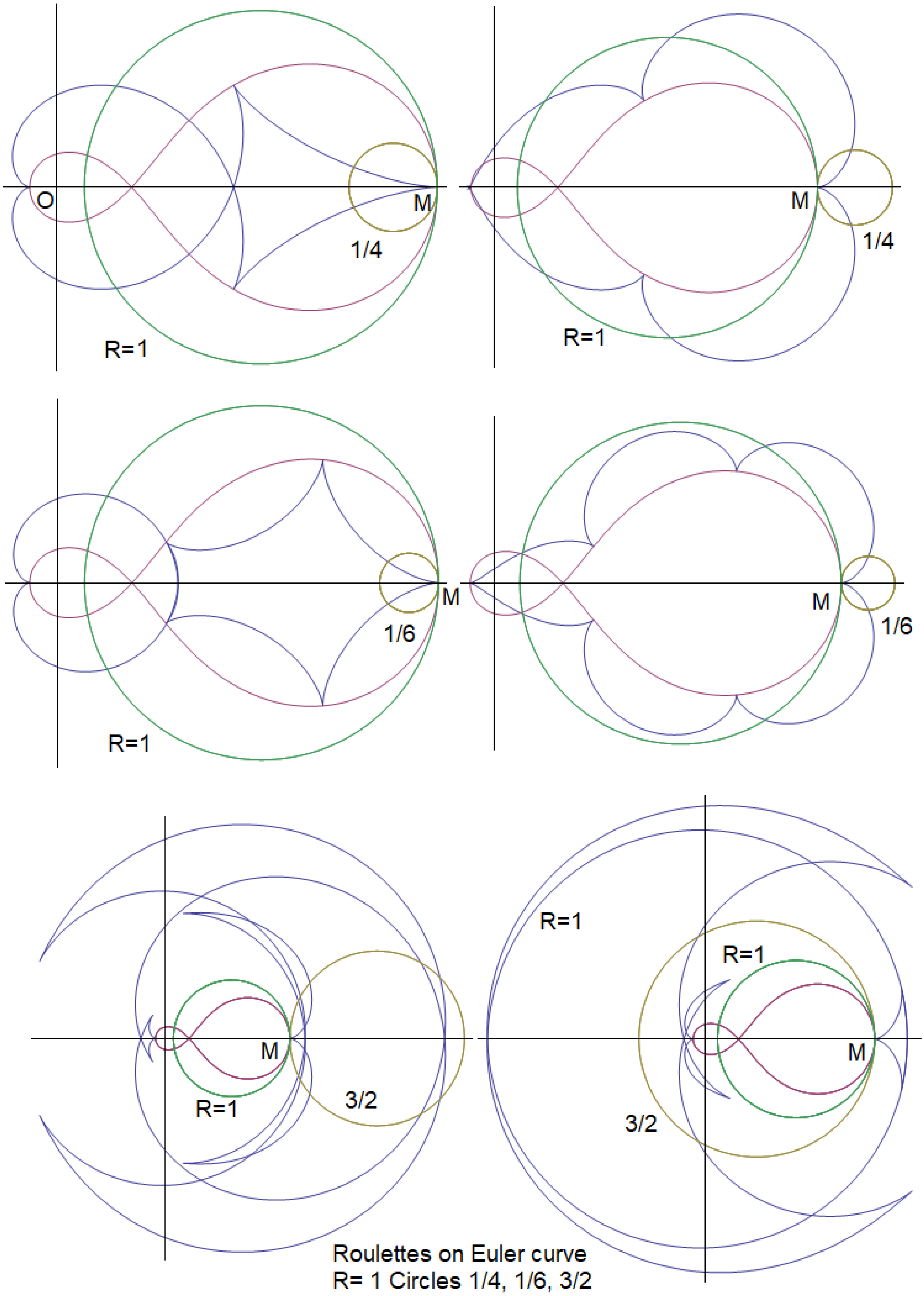

Figure 7: Roulette on Euler curve - rolling curves are circles R=1/4, 1/6 or 3/2.

5 Envelopes of line linked to a circle rolling on another circle in the plane.

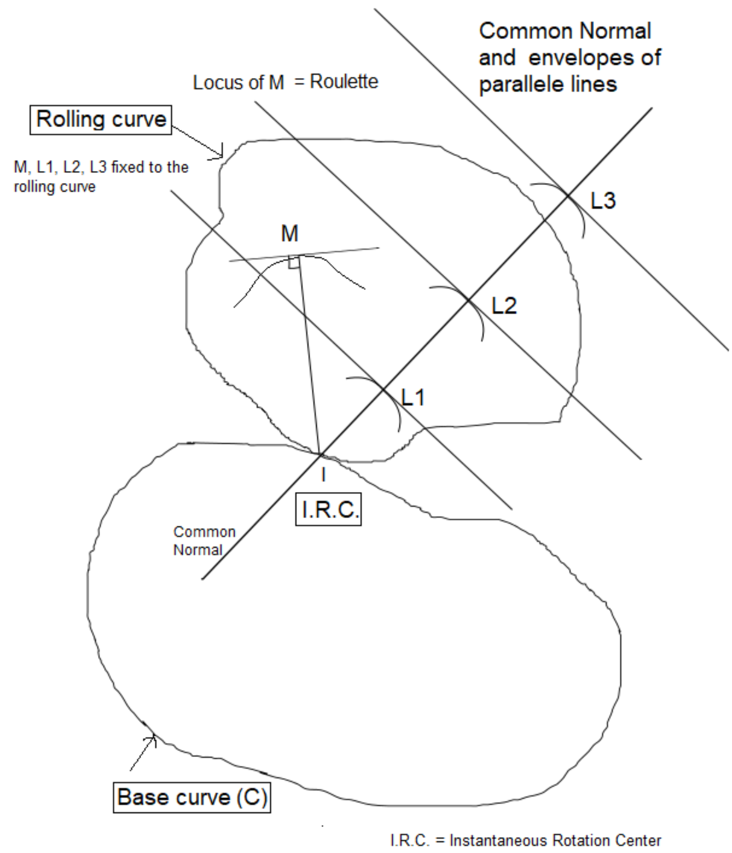

Figure 8: Roulette and envelopes of lines linked to rolling curve

We turn now to a problem of envelopes in relation with epi- or hypo-cycloidals. A fixed plane base circle centered at origin O with radius R and another circle af radius r rolling on the base and we look for the envelope of a line linked to the rolling circle. Since Descartes, we know that the point of contact between this line and its envelope is the orthogonal projection of the instant center of rotation of the motion, the tangent contact between the circles. As for epi- or hypo-cycloidals we distinguish the two cases rolling

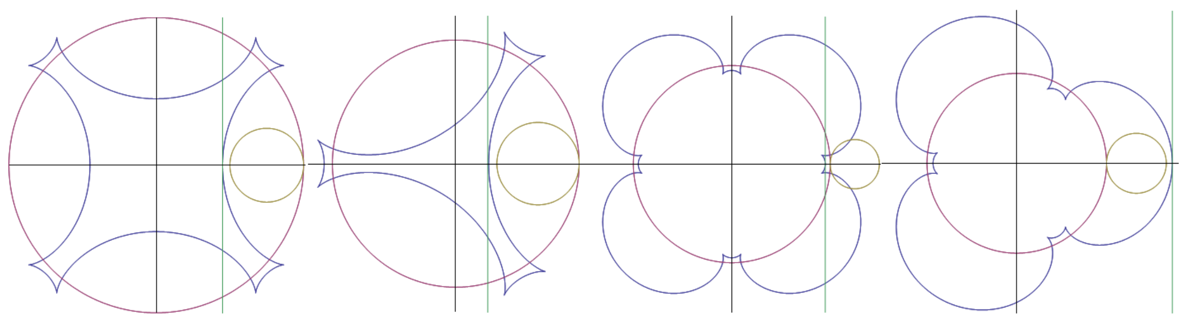

Figure 9: Envelope of a straight line driven by the rolling of a circle R=1/k on a base circle R=1

inside or outside the base circle. These envelopes have a simple relation with classical ponctually generated cycloidals, locus of a point on the rolling circle.

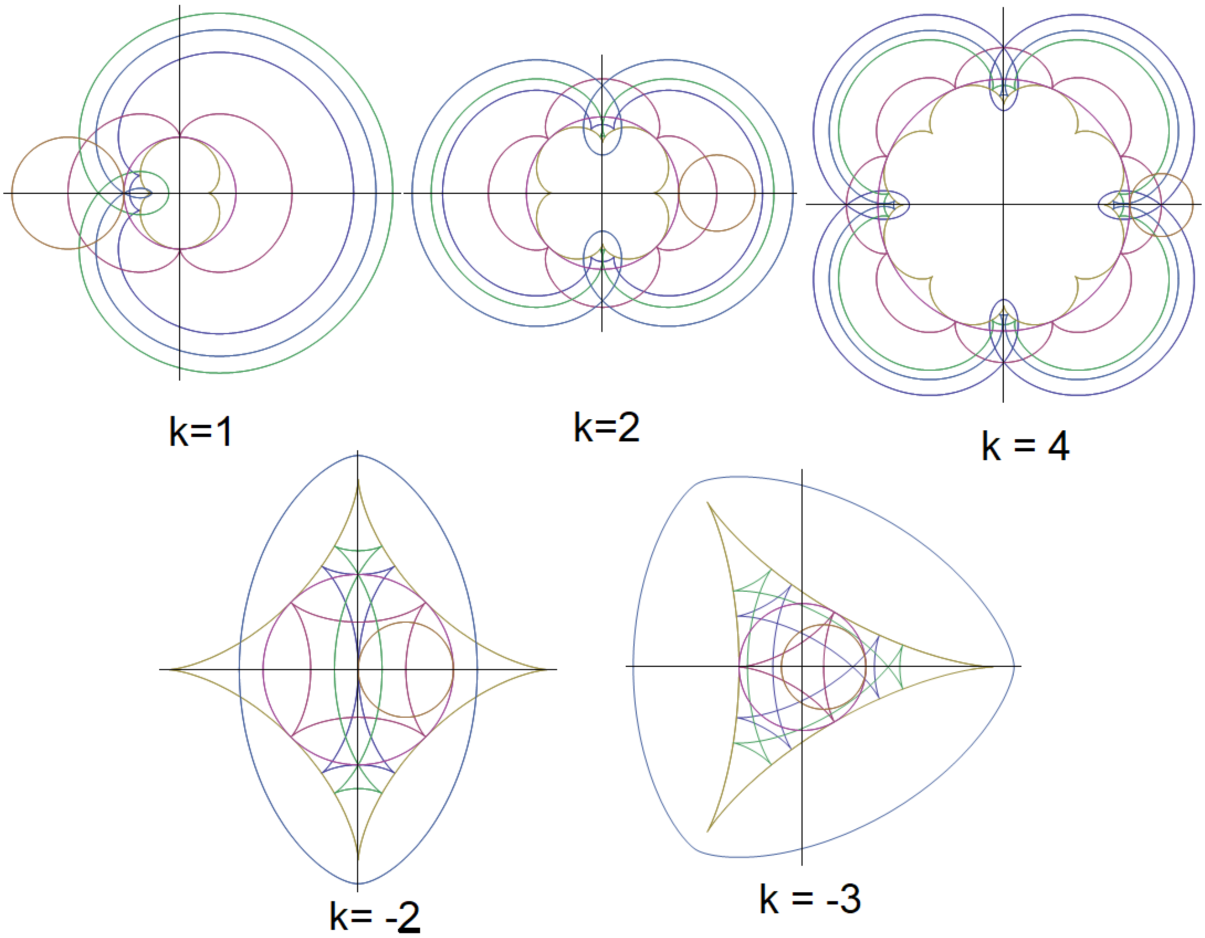

Figure 10: Envelope of a straight line driven by rolling of a circle on another circle.

At the beginning of motion the line is orthogonal to x-axis at distance d from the center of the rolling circle. We set R=1 and $\frac{R}{r} = k$ for Epicycloidals (for hypocycloidals, take: -k) M is the projection of the I the instantaneous center of rotation (ICR) point of tangent contact between base and rolling curves. Some calculations give the equations of the envelope in the two case of rolling, if $k \in N$.

For Epi-cycloidals:

$$\boxed{x_M = \cos t + \frac{1}{k}.(d + \cos(k.t)) \cos(t + k.t)}$$

$$\boxed{y_M = \sin t + \frac{1}{k}.(d + \cos(k.t)) \sin(t + k.t)}$$

For hypo-cycloidals ($k \to -k$) :

$$\boxed{x'_M = \cos t - \frac{1}{k}.(d + \cos(-k.t)) \cos(t - k.t)}$$

$$\boxed{y'_M = \sin t + \frac{1}{k}.(d + \cos(-k.t)) \sin(t - k.t)}$$

These envelopes are indeed involutes of cycloidals with twice numbers of arches. This is obvious that a diameter of the rolling circle has for envelope a cycloidal of the same type with a coefficient k'=2.k with twice the numbers of arches. This is true also for the usual cycloid for a point of a circle rolling on a base line.

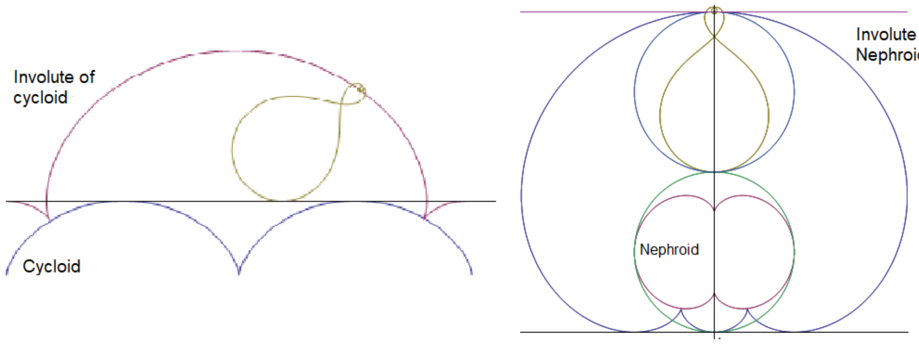

Figure 11: Roulette of inverse of Norwich spiral on a line = Involute of the cycloid and on a circle = Involute of Nephroid.

So we can state the following result :

Property 1 : "The envelope of a line linked to a curve rolling on another fixed curve in the plane, then parallele lines will envelope parallele curves of the roulette"
Since the common normal passes through the Instantaneous Center of Rotation (IRC), the property is almost obvious.

6 Using wheels for circular ground to generate the envelope of a line in cycloidal motion.

The wheels for a circle ground, presented in my paper II, have been studied by Euler and later by J. Serret. These curves may be used to generate by a roulette the involutes of general cycloidals. The envelopes of lines in a rolling motion (a curve rolling on another curve) may be generated by a ponctual roulette. This is the consequence of the dfinition of Gregory's inverse transformation. We consider the wheel associated with the rolling curve and the line and we state the :
Property 2 : "The envelope of a line (Δ) linked to a curve (C_1) rolling on a base curve (C) is

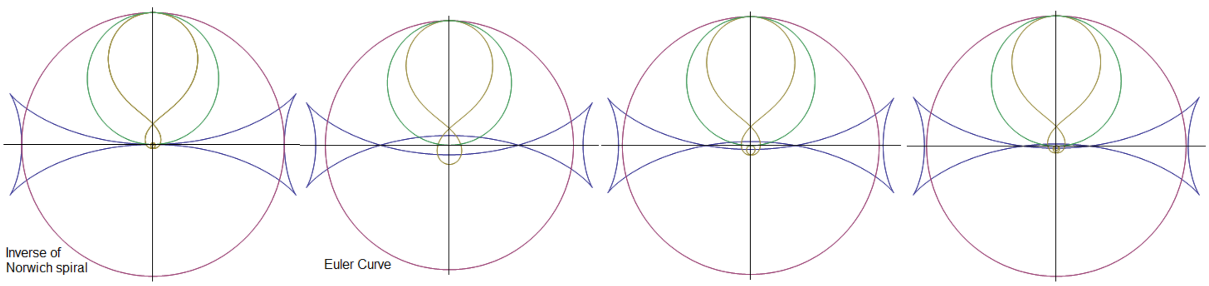

Figure 12: Involutes of astroid as the roulette of pole of some closed wheels for a circle ground on a circle R=1.

also the roulette of pole of the wheel - associated with the couple $[(C_1), (\Delta)]$ - which rolls on the base curve."

There are initial conditions for these motions, the same as for Gregory's transformation. The figures illustrate this property and can be considered as a visual proof.

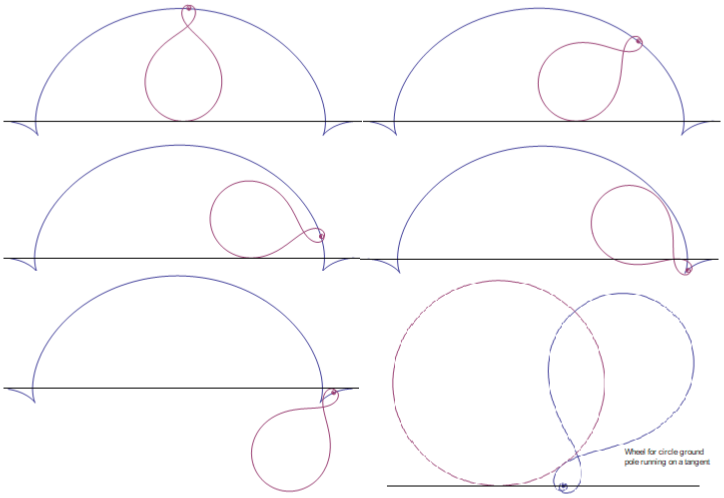

Figure 13: Involute of cycloid as roulette of inverse of Norwich spiral on a line.

References :

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(2) Gomez-Teixeira - Traite des courbes speciales remarquables (1907)
(3) L. Euler : De curvis algebricis, quarum omnes arcus per circulares metiri liceat (1781).

This article is the 28^{th} Some special roulettes.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643 - 1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Part XXVI : A selection of special plane curves $C_k(n,p)$ and a few properties - Cyclodes
Part XXVII : Some plane curves.
Part XXVIII : Some special roulettes and envelopes.

Two papers in french :

1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr