Special cases of $C_k(n,p)$ with explicit expressions of $\rho = f(\theta)$. - Part XXIX -

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Abstract

From the classes of curves $C_k(n,p)$, generalizations of sinusoidal spirals, we extract some special series of curves with explicit polar equation $\rho = f(\theta)$. Curves in $C_1(n,p)$, $C_{-2*}(n,p)$, $C_3(n,p)$ have simple parametric equation with a natural parameter u linked to polar angle θ by a linear relation : $V = u$, $V = \pi/2 - 2u$ and $V = 3u$.

1 Curves with angle $\theta = n\tan(u) + p.u$and V =linear function of u

The curves $C_k(n,p)$ are generalizations of the set sinusoidal spirals : $\rho^n = \sin n\theta$ or $\cos n\theta$. We know that sinusoidal spirals are wheels for the Curves of Ribaucour w.r.t. to the natural base. They defined as the curves with special angle parametrization $\theta = n\tan(u) + p.u$ and a condition on the angle V between vector radius and current tangent.
Among these set of curves we find some special cases when the explicit polar equation $\rho = f(\theta)$ is possible.This happens when p=0 in the parametric expression of the angle and the $\theta = n.\tan u$.

2 The set of cuves $C_1(n,p)$ with $V = u$

These curves have parametric equations :

$$\boxed{\rho = \frac{\sin^{(p+n)}(u)}{\cos^n(u)} \qquad \theta = n\tan(u) + p.u}$$

In this case if p = 0 then $\rho = \tan u^n$ so the polar equation is $\rho = (\frac{\theta}{n})$. This is a parabola spiral general studied by Torricelli and Fermat. This case is in some sense trivial since Mc Laurin transformation is equivalent to a homothety.
These special class of parabolic spirals have parametric equations :

$$\rho = \frac{\sin^n(u)}{\cos^n(u)} = \tan^n u \qquad \theta = n\tan(u)$$

And so :

$$\rho = \left(\frac{\theta}{n}\right)^n = \frac{1}{n^n}.\theta^n = K.\theta^n$$

In the 16ties Gregory of St Vincent, Roberval and others knew the equality of arc of the Archimede Spiral $\rho = a.\theta$ and the Parabola : $y^2 = 2.p.x$ The wheel and its corresponding ground. So the generalized parabola/hyperbola spirals are related to General Parabolas / hyperbolas as wheels and grounds by Gregory's transformation ($y = \rho$, $x = \int \rho.d\theta$).

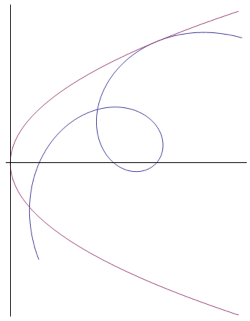

Figure 1: Archimedean spiral rolling on a fixed Parabola

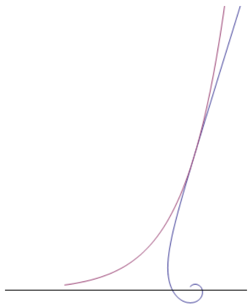

Figure 2: Hyperbolic spiral rolling on a fixedExponential

3 The set of cuves $C_{-2}(n, p)$ with $V = \pi/2 - 2u$

These curves have parametric equations :

$$\boxed{\rho = (\cos u)^{2n}.(\cos 2u)^{(p)} \quad \theta = n \tan u - 2(n+p).u}$$

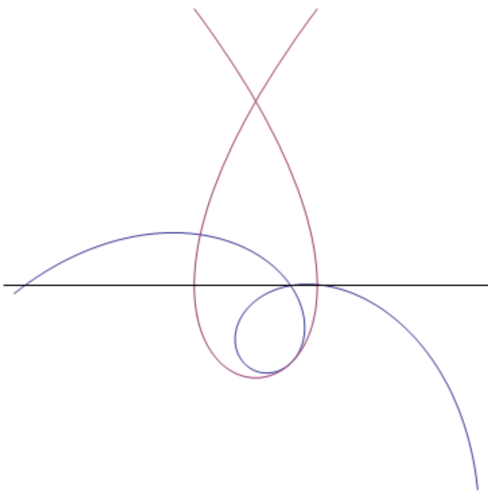

Figure 3: Galileo spiral $\rho = 1 - \theta^2$ rolling on Tschirnhausen's cubic $x = t - t^3/3$, $y = 1 - t^2$

When p = - n then $\theta = n \tan u$ and if we use this value in expression of ρ we get the following serie function of n :

$$\rho = \cos u^{2n} . \cos 2u^{(-n)} = \left[\frac{1}{1-\left(\frac{\sin u}{\cos u}\right)^2}\right]^n = \left[\frac{1}{1-\tan^2 u}\right]^n = \left[\frac{1}{1-(\frac{\theta}{n})^2}\right]^n$$

and by Mc Laurin'stransformation :

$$\rho = \left[\frac{1}{1-(\frac{\theta}{n})^2}\right]^n$$

Here we list some examples :

3.1 $n = 1$: Catalan's curve.

$$\rho = \frac{1}{1-\theta^2}$$

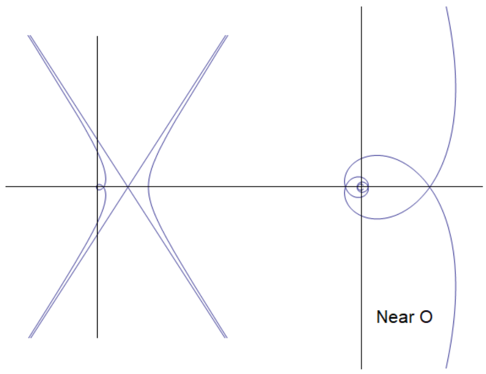

Figure 4: Curve n=1

3.2 $n = 2$: Second curve.

$$\rho = \left[\frac{1}{1-(\frac{\theta}{2})^2}\right]^2$$

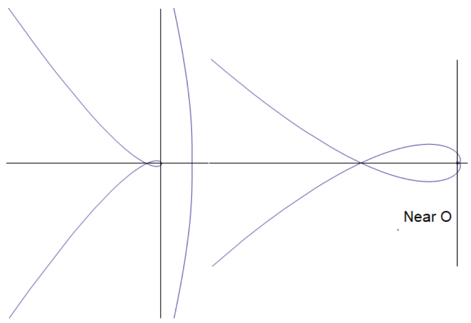

Figure 5: Curve n=2

3.3 $n = 3$: Third curve.

$$\rho = \left[\frac{1}{1-(\frac{\theta}{3})^2}\right]^3$$

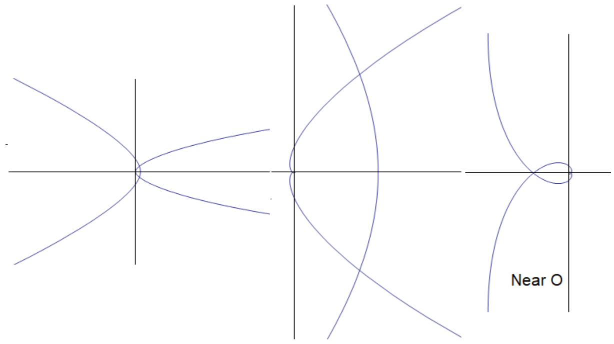

Figure 6: Curve n=3

3.4 $n = -1$: Special Galileo spiral.

$$\rho = 1 - \theta^2$$

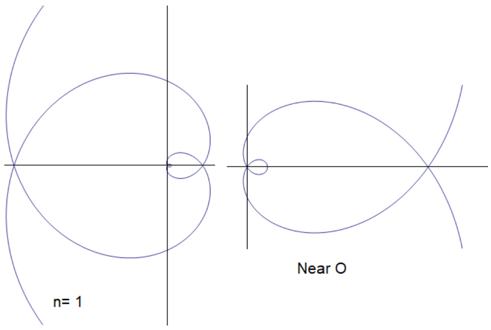

Figure 7: Curve n= -1

3.5 $n = -2$: Second curve.

$$\rho = \left[1 - (\frac{\theta}{2})^2\right]^2$$

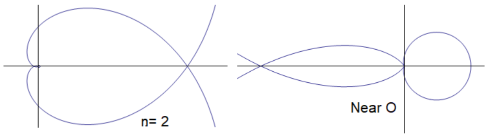

Figure 8: Curve n= -2

3.6 $n = -3$: Third curve.

$$\rho = \left[1 - (\tfrac{\theta}{3})^2\right]^3$$

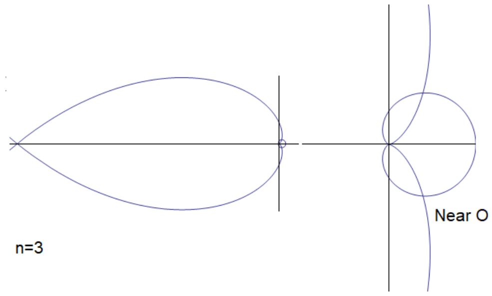

Figure 9: Curve n= -3

4 The set of cuves $C_3(n,p)$ with $V = 3u$.

These curves have parametric equations :

$$\boxed{\rho = \left[\frac{\cos u}{3-\tan^2 u}\right]^n .(\sin 3u)^p \qquad \theta = n.\tan(u) - (n+3p).u}$$

We study the cases $n = -3p$, then term in u is zero and $\theta = -3p.\tan u$:

$$\rho = \left[\frac{\cos u}{3 - \tan^2 u}\right]^{-3p} .\sin^p 3u$$

$$\rho = \left[\frac{3 - \tan^2 u}{\cos^3 u}\right]^{3p} .\sin^p 3u$$

$$\rho = \left[\frac{3 - \tan^2 u}{\cos^3 u}\right]^{3p} .\sin^p 3u = \left[\left(\frac{3 - \tan^2 u}{\cos^3 u}\right)^3 .\sin 3u\right]^p =$$

$$\rho = \left[(3 - \tan^2 u)^4 .\tan u\right]^p$$

Since $\theta = n.\tan u$ then $\tan u = \theta/n$ and we get the Mc Laurin formula :

$$\rho = \left[\left(3 - (\frac{\theta}{n})^2\right)^4 .\left(\frac{\theta}{n}\right)\right]^{\frac{n}{3}}$$

And finally we get, since $\tan u = \frac{\theta}{3p}$ and for convenience we use p as the new exponent $n = \pm 3.p$:

$$\boxed{\rho = \left[\left(3 - (\tfrac{\theta}{3.p})^2\right)^4 .\left(\tfrac{\theta}{3.p}\right)\right]^p}$$

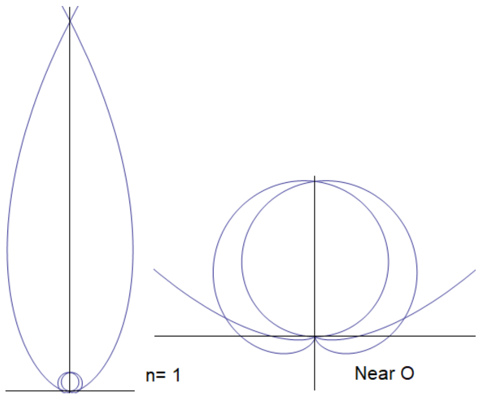

Figure 10: Curve p = 1

4.1 $p = 1$: First curve.

$$\rho = \left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)$$

4.2 $p = 2$: Second curve.

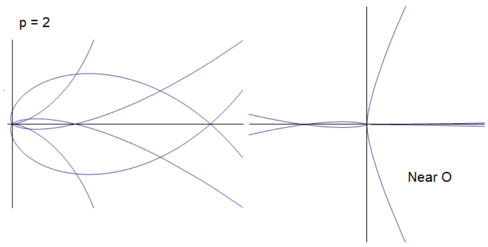

Figure 11: Curve p = 2

$$\rho = \left[\left(3 - \left(\frac{\theta}{6}\right)^2\right)^4 \cdot \left(\frac{\theta}{6}\right)\right]^2$$

4.3 $p = 3$: Third curve.

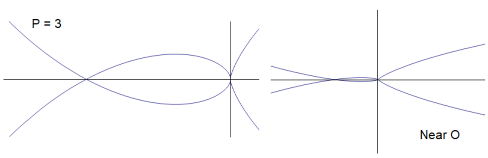

Figure 12: Curve p=3

$$\rho = \left[\left(3 - \left(\frac{\theta}{9}\right)^2\right)^4 \cdot \left(\frac{\theta}{9}\right)\right]^3$$

4.4 $p = -1$: First curve

$$\rho = \frac{1}{\left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)}$$

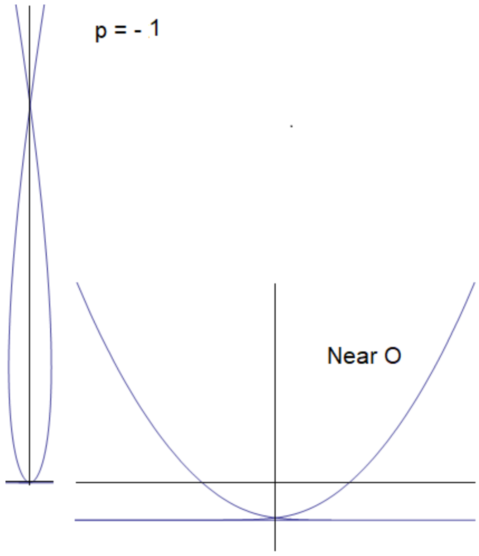

Figure 13: Curve p = 3

4.5 $p = -2$: Second curve.

$$\rho = \frac{1}{\left[\left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)\right]^2}$$

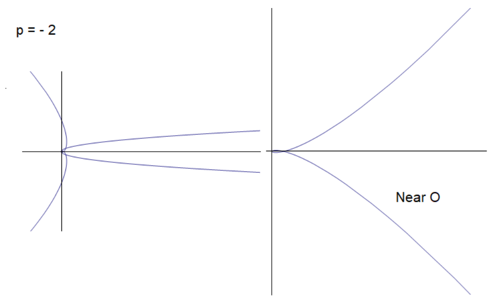

Figure 14: Curve p= -2

4.6 $p = -3$: Third curve.

$$\rho = \frac{1}{\left[\left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)\right]^3}$$

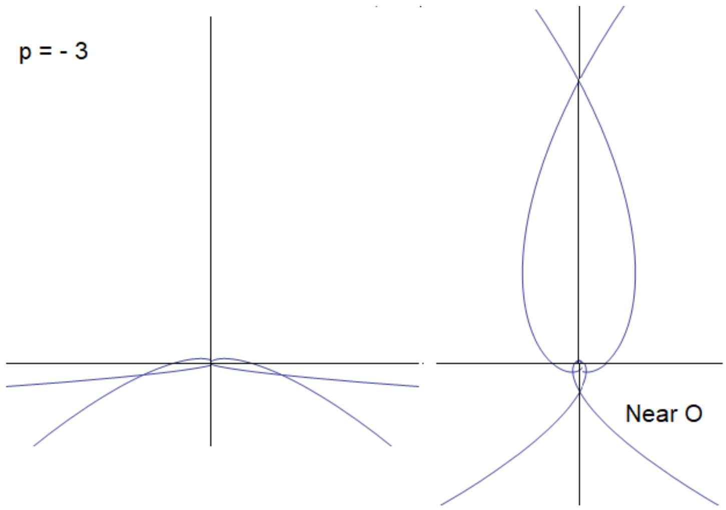

Figure 15: Curve p= -3

4.7 Curves with explicit $\rho = f(\theta)$ expressions.

The curves presented in the three classes above, associated with the $C_k(n, p)$sets of curves, have a simple property for the angle V is equal to a multiple of the parameter u of polar angle $\theta = n.\tan u$. But it is not easy to find geometric properties of these curves.
We give here two special cases of couples of associated wheel/ground.

4.7.1 Wheel is the Second curve of serie $V = \pi/2 - 2t$ for p = -2

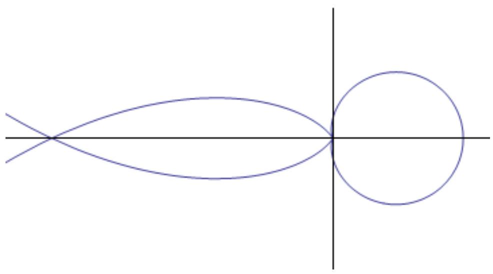

Figure 16: Curve n= -2

$$\rho = \left[1 - \left(\frac{\theta}{2}\right)^2\right]^2$$

The parametric equations of the ground are :

$$x = 16.\theta - \frac{8}{3}.\theta^3 + \frac{1}{5}.\theta^5 \qquad y = (4 - \theta^2)^2$$

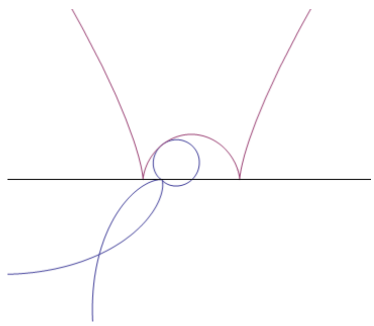

Figure 17: Curve C2(n,-n) n= -2 as the wheel

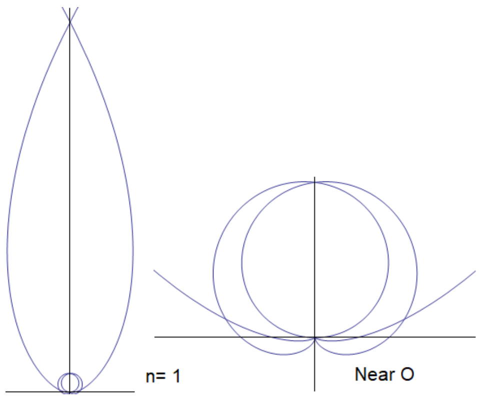

Figure 18: Curve n= -2

4.7.2 Wheel is the first curve of serie V=3u for p=1

$$\rho = \left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)$$

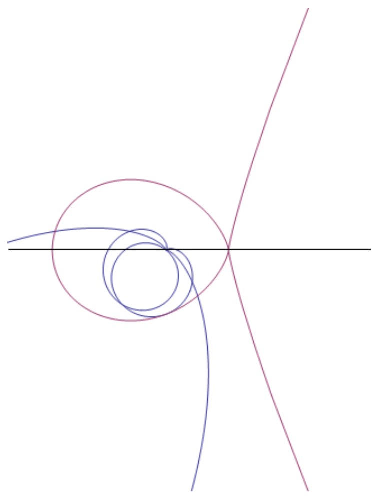

Figure 19: Curve C3(- 3.p, p) p = 1 as the wheel

The parametric equations of the ground(red curve above) are :

$$x = \frac{(\theta^2 - 27)^5}{196830} \qquad y = \left(3 - \left(\frac{\theta}{3}\right)^2\right)^4 \left(\frac{\theta}{3}\right)$$

References :

(1) A - H. Brocard , T. Lemoine - Courbes geometriques remarquables Blanchard Paris (1967)
(2) Gomez-Teixeira - Traite des courbes speciales remarquables (1907)

This article is the 29^{th} Special cases of $C_k(n,p)$.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A historical tale of a question that raised at the beginning of the calculus (1643 - 1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Part XXVI : A selection of special plane curves $C_k(n,p)$ and a few properties - Cyclodes
Part XXVII : Some plane curves.
Part XXVIII : Some special roulettes and envelopes.
Part XXIX : Special cases of $C_k(n,p)$.
Part XXX : The Quintic of L'Hopital
Two papers in french :

1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
Gregory's transformation on the Web : http://christophe.masurel.free.fr