# The Quintic of L'Hopital - Part XXX - 

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#### Abstract

We present some properties of the Quintic of l'Hopital found by the marquis de l'Hopital in 1700 as a special solution of a mechanical problem posed by Johann Bernouilli in 1696. This curve has similarities with the Tschirnhausen's Cubic.


## 1 The Quintic of L'Hopital :

In 1695 Johann Bernoulli in a letter to Leibniz and in 1696 in Acta Eruditorum presented a question :
"To find a plane curve with constant reaction such that if a particle descends along it by the pull of gravity (the gravitational field is supposed to be uniform), then the reaction of the curve on the particle has a constant intensity; conversely, the force applied by the particle on the curve has a constant intensity."
The Marquis of l'Hopital presented a general solution for the problem and, as a special solution, his Quintic in 1700 . It is a curve of direction and has some resemblance with Tschirnhausen's cubic. The parametric equations of Quintic of l'Hopital in the $(x, y)$-plane are :

$$
x=2\left(t-\frac{t^{5}}{5}\right) \quad y=\left(1+t^{2}\right)^{2}
$$



Figure 1: Quintic of L'Hopital

## 2 A physical problem : the curve of constant pressure.

We give the solution of L. Lecornu (1904 BSMF) for the above problem of Johann Bernoulli. The moving point has a mass equal to unity. g is the gravity, $\theta$ the angle between tangent and x-axis, v the velocity, $R_{c}$ the radius of curvature, and sthe curvilinear abcissa. We note $R_{c}=\lambda . g$ the force of the curve on the mobile.
The centrifugal force is :


Figure 2: Bernoulli's physical problem

$$
\begin{equation*}
\frac{v^{2}}{R_{c}}=g \cdot(\lambda+\cos \theta) \tag{1}
\end{equation*}
$$

The theorem of cinetic energy is :

$$
\begin{equation*}
v \cdot d v=g \cdot \sin \theta \cdot d s \tag{2}
\end{equation*}
$$

We have $R_{c}=d s / d \theta$, elimination of $R_{c}$ and ds gives :

$$
\begin{equation*}
\frac{d v}{v}=\frac{\sin \theta d \theta}{\lambda+\cos \theta} \tag{3}
\end{equation*}
$$

By integration we get :

$$
\begin{equation*}
v=\frac{k}{\lambda+\cos \theta} \tag{4}
\end{equation*}
$$

We put this value in (1) so :

$$
\begin{equation*}
R_{c}=\frac{k^{2}}{g(\lambda+\cos \theta)}=\frac{v^{2}}{k g} \tag{5}
\end{equation*}
$$

The radius of curvature is therefore proportional to the cube of the speed. Let $y$ be the distance from the mobile to the horizontal over which its speed would cancel out. We have :

$$
\begin{equation*}
y=\frac{v^{2}}{2 g}=\frac{k^{2}}{2 g(\lambda+\cos \theta)} \tag{6}
\end{equation*}
$$

Comparing (5) and (6) gives :

$$
R_{c}=\frac{2 \sqrt{2} g}{k} y^{3 / 2} .
$$

Equation (4) shows that the hodograph of the movement is a conical section with excentricity e.

$$
\begin{array}{cl}
v=\frac{a\left(1-e^{2}\right)}{1+e \cos \theta} \\
e=1 / \lambda \quad & a=\frac{k e}{1-e^{2}}=\frac{k \lambda}{\left(\lambda^{2}-1\right)}
\end{array}
$$

There are three kinds of curves depending on the shape of the hodograph $e=1 / \lambda$ :

1) Hodograph is an ellipse $(\lambda>1)$ the parametric equations are :

$$
\begin{gathered}
y=\frac{v^{2}}{2 g}=\frac{a^{2}}{2 g}(1-e \cos u)^{2} \\
x=\frac{a^{2} e}{g \cdot \sqrt{1-e^{2}}}\left(-\frac{3}{2} e u+\left(1+e^{2}\right) \sin u-\frac{e}{4} \sin 2 u\right) \\
\left.s=\frac{a^{2} e}{g \cdot \sqrt{1-e^{2}}}\left[\left(1+\frac{e^{2}}{2}\right) u-2 e \sin u+\frac{e^{2}}{4} \sin 2 u\right)\right]
\end{gathered}
$$

1) Hodograph is an hyperbola $(\lambda<1)$ so :

$$
\begin{gathered}
y=\frac{a^{2}}{2 g}\left(1+\frac{e^{2}}{2}-2 . e \cosh t+\frac{e^{2}}{2} \cosh 2 t\right) \\
x=\frac{a^{2} e}{g \cdot \sqrt{e^{2}-1}}\left(-\frac{3}{2} e \cdot t+\left(1+e^{2}\right) \sinh t-\frac{e}{4} \sinh 2 t\right) \\
t-e \cdot \sinh t=\frac{\sqrt{e^{2}-1}}{a e} g \cdot t
\end{gathered}
$$

3) The case $\mathrm{e}=1$ a parabolic hodograph corresponds to the case of the Quintic of L'Hopital :

$$
\begin{aligned}
x & =\frac{k^{2}}{4 g}\left(t-\frac{t^{5}}{5}\right) \\
y & =\frac{k^{2}}{8 g}\left(1+t^{2}\right)^{2} \\
t & +\frac{t^{3}}{3}=2 g \cdot t
\end{aligned}
$$

Among the infinite number of solutions of this physical problem there is an algebraic solution called the Quintic of L'Hopital Other formulation of the question :
The evolutes of such curves are the solution of the following problem (also posed by Jean Bernoulli) : determining a curve on which to wind the wire of a pendulum so that the tension of the wire of this pendulum remains constant.

### 2.1 Focus of L'Hopital Quintic

The foci of an algebraic plane curve are points of intersection of two tangents from cyclic points I or J. In parametric coordinates : $x=f(t), y=g(t)$, the circular lines through the foci are the tangents at the points given by $f^{\prime}(t)= \pm i . g^{\prime}(t)$. For L'Hopital Quintic : $2\left(1-t^{4}\right)= \pm i .4 . t\left(1+t^{2}\right)$ and we get $(t \pm i)^{2}=0$ and the value of for point of tangency of the two isotropic lines are $t= \pm . i$. So :

$$
x+i y=2 i\left(1-i^{4} / 5\right)+i\left(1+i^{2}\right)^{2}=a+i b=0+i .8 / 5
$$

So the coordinates of the focus F are $(0,8 / 5)$.

### 2.2 Double points of L'Hopital Quintic

These double points are on the y -axis of symmetry for $\mathrm{x}=0$. So for t double root cancelling the x-coordinate :

$$
x=2\left(t-\frac{t^{5}}{5}\right)=0
$$

We find $t= \pm \sqrt{5}$ and $t= \pm i \sqrt{5}$.
The first value gives the evident point $D(0,6+2 \sqrt{5})$.
The second value gives an isolated double point $I(0,6-2 \sqrt{5})$. Near and under the focus F .

## 3 Equation w.r.t. the focus and the directrix



Figure 3: Equation (MF/MH) or $(\rho, y)$.
The L'Hopital Quintic ( $\rho, \mathrm{y}$ ) can be expressed in the form $\mathrm{f}(\mathrm{MF}, \mathrm{MH})=0$ see (4) : distance $\rho$ to the focus and y to the directrix, the same used since antiquity for conics. The directrix is the axis x'x and is the chord of contact of the two circular lines through focus F (so the directrix corresponding to F).

$$
\begin{gathered}
y=\left(1+t^{2}\right)^{2}=M H \\
\rho^{2}=\left(x^{2}+(y-8 / 5)^{2}=M F^{2}\right.
\end{gathered}
$$

It can be verified that the equation focus-directrix $(\rho, y)$ is :

$$
25 \rho^{2}=y^{2}(4 \sqrt{y}+5)
$$

## 4 Cartesian and polar equations of the Quintic of L'Hopital

We need the equation of the L'Hopital Quintic with vertical axis in an orthonormal frame in the following form :

$$
x=2\left(t-\frac{t^{5}}{5}\right) \quad y=\left(1+t^{2}\right)^{2}
$$

and :

$$
\tan V=\frac{d x}{d y}=\frac{1-t^{2}}{2 t}=\tan (\pi / 2-2 u)
$$

The parameter $t=\tan u$ implies $V=\pi / 2-2 u$. The polar equation with the pole at the focus of the Quintic of L'Hopital is :

$$
\begin{gathered}
25 . r^{2}=y^{2} \cdot(4 \sqrt{y}+5)=\left(1+t^{2}\right)^{2} \cdot\left(4\left(1+t^{2}\right)+5\right) \\
\tan \theta=\frac{\Delta x}{\Delta y}=\frac{2\left(t-t^{5} / 5\right)}{\left(1+t^{2}\right)^{2}-8 / 5}
\end{gathered}
$$

The curve has a symmetry w.r.t. Oy-axis. Three horizontal lines in the plane of L'HQ play a special role in the geometry of the curve :

- Directrix/natural base-line ( $\mathrm{y}=0$ ),
- Tangent to the summit $\mathrm{S}(0,1)$ for $\mathrm{u}=0$ is $(\mathrm{y}=+1)$ and
- Double normal $(y=+4)$.


## 5 A subclass of curves linked with wheels $C_{2 *}(n, p)$ and to "curves of direction" as a generalization of Tschirnhausen's cubic or Nephroid :

We study grounds corresponding to wheels for which $p=0$ with parametric equations :

$$
\rho=(\cos t)^{2 n} \quad \text { and } \quad \theta=n(\tan t-2 t)
$$

We use these wheels in polar to find ground curves using direct Gregory's transformation ( $y=\rho$ and $\mathrm{x}=\int \rho . d \theta$ ) by one integration we get the parametric equation of the ground in the plane $(x, y)$ for n positive integer :

$$
y=\rho=\cos ^{2 n} t \quad \text { and } \quad x=\int \rho \cdot d \theta=n \int \cos ^{2 n} t \cdot\left(\tan ^{2} t-1\right) \cdot d t
$$

We identify these equations with the one of general caustics generated by the arbitrary functions $f(t)$ and $g(t)$ which are COD when algebraic. We find :

$$
f(t)=2 n \cos ^{2 n-1} t \sin t \quad \text { and } \quad g(t)=\tan t
$$

## 6 Ground corresponding to wheels $C_{2 *}(n, 0)$ when $n>0$

The grounds corresponding to the class of wheels $C_{2 *}(\mathrm{n}, \mathrm{p})$ are caustics by reflection in the same way as for the Nephroid. The subclass of grounds corresponding to curves $C_{2 *}(\mathrm{n}$, 0 ) - with $2 . n \in Z$ - are caustics of plane curves.
we shall consider the curves for small $n$ positive and negative. Since the ordinate $y=$ $\cos ^{2 n} t$ we explore half integers (so : 2.n=an integer) and examine two classes $y<1$ so n is positive and $y>1$ for negative n . The first cases are :
$\mathrm{n}=1 / 2$ : Poleni's curve,
$\mathrm{n}=1$ : Circle,
$\mathrm{n}=3 / 2$ : Nephroid,
$\mathrm{n}=-1$ : Tschirnhausen's Cubic,
$\mathrm{n}=-2$ : Quintic of l'Hopital .


Figure 4: Ground for $C_{2 *}(n, 0)$ Angle $V=\pi / 2-2 u$ from $\mathrm{n}=1 / 2$ to 4 by step $1 / 2$

The following curves: $C_{* 2}(n, 0): \rho=\cos ^{2 n} t, \quad \theta=n(\tan t-2 t)$ presents interesting properties : they are wheels for caustics by reflection. The serie of curves that we discuss here is a subserie of above grounds for the wheels $C_{2 *}(n, p)$ when $\mathrm{p}=0$ and $\mathrm{n}=1 / 2$ to 4 step $1 / 2$ :


Figure 5: Ground for $C_{-2}(n, 0)$ Angle $V=\pi / 2-2$ from $\mathrm{n}=-1$ to -4 by step $-1 / 2$
We giveabove the the graphs of curves when $\mathrm{p}=0$ and $\mathrm{n}=-1 / 2$ to -4 step $-1 / 2$ are algebraic with a rational arc length and are curves of direction stricto sensu. For $n$ half integer the curves are not algebraic so not curves of direction stricto sensu.

## 7 Properties derived from the system ground/wheel

In this section we use Gregory's transformation and three base-lines $\mathrm{y}=0$ (the directrix), $\mathrm{y}=+1$ tangent to S and $\mathrm{y}=+4$ (the double normal) and the reverse Gregory's transformation ( $G T^{-1}$ ).

### 7.1 Wheel for directrix : $y=0$

We search for the wheel in parametric polar coordinates and for the first one ( $y=0$ ) we find :

$$
\rho=y=\left(1+t^{2}\right)^{2}=\frac{1}{\cos ^{4} u} \quad \text { and } \quad \theta=2 \tan (u)-4 u
$$

It is a $C_{2 *}(n, p)$ for $\mathrm{n}=2, \mathrm{p}=0$.
The antipedal of this wheel is

$$
\rho=\frac{1}{\cos ^{4} u \cos 2 u} \quad \text { and } \quad \theta=2 \tan (u)-6 u
$$

It is a $C_{2 *}(n, p)$ for $\mathrm{n}=2, \mathrm{p}=-1$.
The antipedal of this wheel when rolling on x'Ox generates the Quintic of L'Hopital .

### 7.2 Wheel for line : $y=+4$

For the other base-line $(\mathrm{y}=+4)$ - the double normal - and the same L'HQ the wheel is :

$$
\rho=\left(t^{2}+3\right)\left(t^{2}-1\right) \quad \text { and } \quad \theta=\frac{4}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}}-2 t
$$

These two wheels can roll one on the other around two poles at the distance of 3 with


Figure 6: Wheel for the Quintic of L'Hopital for Directrix $\mathrm{y}=0$ and double normal $\mathrm{y}=+4$ usual conditions.

### 7.3 Wheel for tangent to $\mathrm{S}: \mathrm{y}=+1$

Using Gregory's transformation: $\rho=\left(1+t^{2}\right)^{2}-1$ and $\theta=\int \frac{d x}{y}=\int \frac{2\left(1-t^{4}\right) d t}{2 t^{2}+t^{4}}$.

$$
\rho=\left(1+t^{2}\right)^{2}-1 \quad \text { and } \quad \theta=-1 / t-2 t+\frac{3}{\sqrt{2}} \arctan \frac{t}{\sqrt{2}}
$$

When the Quintic of L'Hopital rolls along the xx ' axis the pole/focus describes the curve:

$$
x=2\left(t-\frac{t^{5}}{5}\right) \quad y=\left(1+t^{2}\right)^{2}-4
$$

## 8 L'Hopital Quintic as the caustic of a plane curve

Tschirnhausen's cubic is the caustic by reflection on the parabola or on the semicubic parabola. By analogy there are two plane mirror-curves for the Quintic of L'Hopital. We recall her some results of my paper No 25 .


Figure 7: Wheel of L'Hopital Quintic-ground for tangent at S.



Figure 8: Wheel rolling on the Quintic of L'Hopital base line is the tangent at $\mathrm{S}: \mathrm{y}=+1$


Figure 9: Wheel rolling on L'Hopital Quintic base line is double Normal $\mathrm{y}=+4$

### 8.1 Curve of direction for $\mathbf{n}=-2$ : Quintic of L'Hopital or Looping curve.

This second curve is L'Hopital curve (see fig.10) :

| Curve | Mirror 1 | Mirror 2 |
| :--- | :--- | :--- |
|  | $X_{M}=1+\frac{2 t^{2}}{3}+\frac{t^{4}}{5}$ | $X_{M^{\prime}}=1-\frac{t^{2}}{3}\left(6+t^{2}\right)$ |
|  | $Y_{M}=-\frac{4}{15} t\left(5+t^{2}\right)$ | $Y_{M^{\prime}}=\frac{4}{15} t^{3}\left(5+t^{2}\right)$ |
| $n=2$ | Anti-bisectant-Involute | Caustic $=$ curve of direction |
|  | $X_{D}=\frac{\left(15-15 t^{2}+5 t^{4}+3 t^{6}\right)}{\left(15+15 t^{2}\right)}$ | $X_{C}=\left(1+t^{2}\right)^{2}$ |
|  | $Y_{D}=\frac{8}{15} t^{\left(\frac{3}{2} \frac{\left.1+t^{2}\right)}{\left(1+t^{2}\right)}\right.}$ | $X_{C}=\frac{2}{5} t\left(\left(t^{4}-5\right)\right)$ |



Figure 10: Quadruplet $\mathrm{n}=-2$ : Two Mirrors-Bissectant and Quintic of L'Hopital

## 9 The evolute of the Quintic of L'Hopital

The evolute (fig.11) of the Quintic of L'Hopital has algebraic parametric equations in x an $y$ and is a also a curve of direction (Laguerre) with a rational arc length :

$$
x=4 t^{3}\left(1-\frac{3}{5} t^{2}\right) \quad y=\left(1+t^{2}\right)^{2}\left(2-t^{2}\right)
$$

The arc length is $d s=6 t\left(1+t^{2}\right)^{2} . d t$ so $s=\left[3 t^{2}+3 t^{4}+t^{6}\right]_{t o}^{t 1}$.
References :
(1) Johann Bernoulli Letter to Leibniz (january 1695).
(2) Johann Bernoulli Acta Eruditorum, Supplement II, 1696 p 291.
(3) G. De l'Hopital, Solution d'un probleme physico-mathematique. Memoires de mathematique et de physique tires des registres de l'Academie royale des Sciences (Paris) de 1700, p. 9-21.
(4) E. Turriere - La courbe de L'Hopital. L'enseignement des mathematiques 36 (1937).
(5) E. turriere Notes sur des courbes speciales algebriques. Anois da Faculdade de Ciencias do Porto, t. XX, 1936.


Figure 11: Evolute of the Quintic of L'Hopital
(6) L. Lecornu Sur le mouvement dun point pesant guide par une courbe rigide. Bulletin de la S.M.F., tome 32 (1904), pp. 50-56.
(7) https://mathcurve.com/courbes2d.gb/quintique de l'hopital

This article is the $X X X^{\text {th }}$ part on Gregory's transformation and related topics.
Part I : Gregory's transformation.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves. Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX: Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuides, causticoides.
Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.
Part XXI : Curves of direction, minimal surfaces and CPG duality.
Part XXII : Equality of arc length of the parabola and the Archimede spiral.A histori-
cal tale of a question that raised at the beginning of the calculus (1643-1668) Hobbes, Roberval, Mersenne, Torricelli, Fermat, Pascal and J. Gregory.
Part XXIII : Rectangular hyperbola - Circle Geometric properties and formal analogies.
Part XXIV : Angular relations defining curves - Sectrices of Maclaurin - Plateau's curves.
Part XXV : Caustic by reflection and curves of direction - looking for examples.
Part XXVI : A selection of special plane curves $C_{k}(n, p)$ and a few properties - Cyclodes Part XXVII : Some plane curves.
Part XXVIII : Some special roulettes and envelopes.
Part XXIX : Special cases of $C_{k}(n, p)$ with explicit expressions of $\rho=f(\theta)$.
Part XXX : The Quintic of L'Hopital.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr

