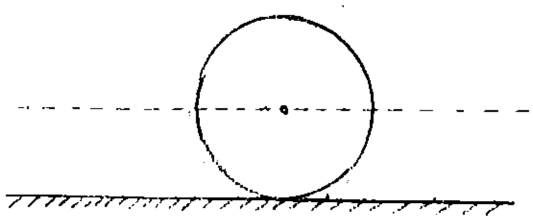
A GENERALIZATION of the WHEEL
or the ADAPTABLE WHEEL

The wheel, whose inventor is unknown, seems to be one of those great ideas that never cease to meet new applications. It is fascinating to note that the principle of the wheel has not changed for about four thousand years. During that time, engineers have constructed roads, bridges and tunnels, and modified the earth so that the wheel could reach previously unaccessible points.

Its great success comes essentially from its facility of construction in addition to its simplicity. Heretofore built on the model of the disc with the hub fixed at the center and the circle making the contact with the ground (fig. 1), the wheel is scarcely flat on the earth, hence we have had to build thoroughfares and adapted the ground to the wheel.

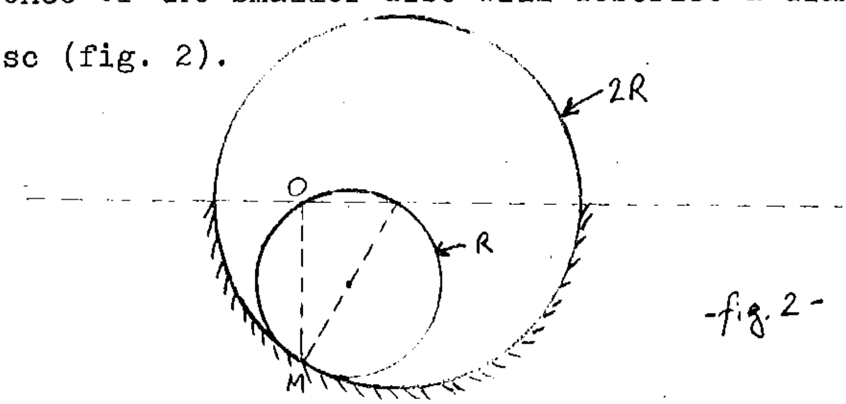


-fig 1-

The following article will show how it is possible to make the inverse adaptation; that is the one of the wheel to the ground. Although this idea may at first appear fanciful, it has some striking applications. Animals move on the ground with the help of articulated limbs, while the wheel has facilitated man's terrestrial mobility. The adaptable wheel, that fits perfectly to the ground that it crosses, would represent the missing link between these two distinct technical systems. Just as the Trolley is a middle course between the train and the bus, so it would be useful to assimilate the adaptable wheel to a kind of articulated limb system.

The history of mathematics gives examples of the theory which is of interest in resolving the above proposition. Copernic and Cardan (16th century) have given this theorem (sometimes called Lahire's gear in France):

- If a disc of radius R rolls, without slipping, along the inner edge of a disc of double its radius (2R) then a point O fixed on the circumference of the smaller disc will describe a diameter of the larger disc (fig. 2).



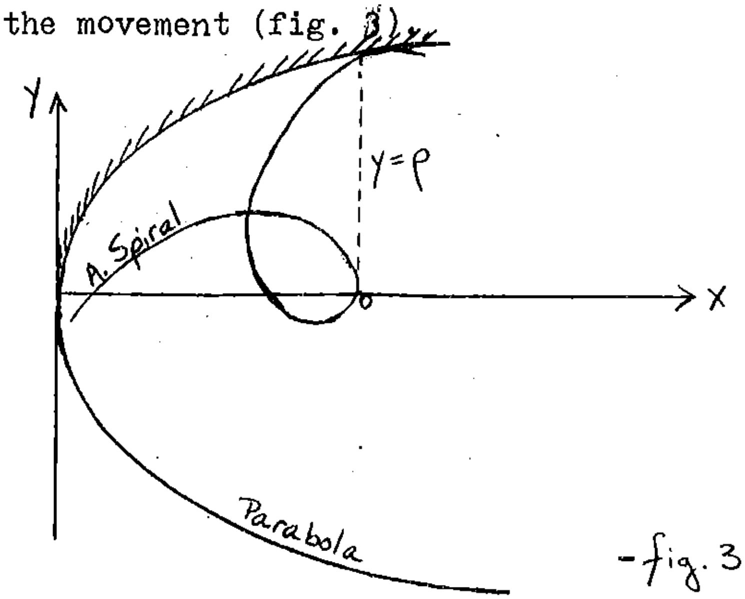
-fig. 2-

Note: All the examples in this article are in the plane

A little later, in the 17th century, Cavalieri, Torricelli, Grégoire de St. Vincent and many others noticed, through the reading of Archimede's work, a curious similarity between Archimede's spiral (equation: $\rho = a \cdot \theta$ in polar coordinates) and the parabola (equation: $y^2 = 2ax$ in rectangular coordinates). The arc lengths of the two curves are equal between the points $y = \rho = 0$ and the current point M, $y = \rho$

$$\left[s_{spiral} \right]_{\rho=0}^{\rho=\rho} = \left[s_{parabola} \right]_{y=0}^{y=\rho}$$ and the area enclosed

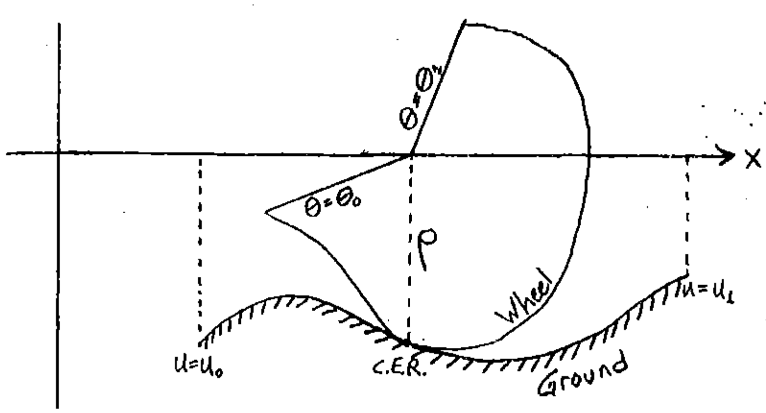
between the x-axis, the parabola and a line parallel to the y-axis at the end of the arc is twice that of the corresponding spiral sector. We can show that it is possible to roll the spiral on the inner edge of the parabola so that the pole 0 of the spiral describes the x-axis, the parabola's axis of symetry. The summits are in coincidence at the beginning of the movement (fig. 3).

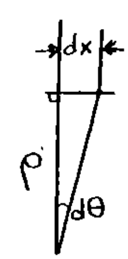


-fig. 3-

These two simple examples allow us to express the problem in general terms. Let us take a ground simulated by any plane curve (x(u), y(u) in rectangular coordinates - u is a parameter) we shall now search for the equation in polar coordinates (ρ,θ) of the adapted wheel so that it may roll, without sliding, on the ground in such a way that the hub describes a horizantal line (which will be supposed to be the x-axis).

so: $u_0 \leq u \leq u_1$ and $\theta_0 \leq \theta \leq \theta_1$





The Center of Elementary Rotation (C.E.R.) of the wheel is the point of contact with the ground which is on the vertical line of the point (i.e. normal to the x-axis). So we can write: $\rho = y$ (1)

By an elementary rotation about the C.E.R. we obtain: $dx = \rho\, d\theta$ (2)

These equations give the profile of the ground or of the wheel if one of them is known:

A) If we know the wheel: $\rho = f(\theta)$

then: $\begin{cases} \rho = y \\ x - x_0 = \int_{\theta_0}^{\theta} \rho \cdot d\theta \end{cases}$ We thus obtain the parametric equation of the ground (θ is equivalent to u) in rectangular coordinates.

B) If we know the ground: $x(u)$, $y(u)$

then: $\begin{cases} \rho = y \\ \theta - \theta_0 = \int_{u_0}^{u} \frac{dx}{y} \end{cases}$ Hence yielding the parametric equation of the wheel in polar coordinates. (Of course, there is a condition: $y \neq 0$ on the interval $[u_0, u_1]$.)

<u>Note</u>: As in any problem concerning differential equations, initial conditions must be verified: $\rho = \rho_0$ and $\theta = \theta_0$. These conditions define the position of the wheel at the beginning of the movement. In all the examples of this article the conditions are assumed to be verified.

This problem has been investigated by many mathematicians during the 17th century but the one who went the furthest in the research by geometric means was J. Gregory (1638-1675). We shall call the transformation (wheel-ground) Gregory's Transformation, G.T., ($G.T.^{-1}$ being the inverse which gives the wheel's form, knowing the shape of the ground).

The Gregory Transformation defined as:

$$\rho = y \quad (1) \qquad \text{and} \qquad \rho \cdot d\theta = dx \quad (2)$$

allows the derivation of many fundamental properties for the correspond-ing curves:

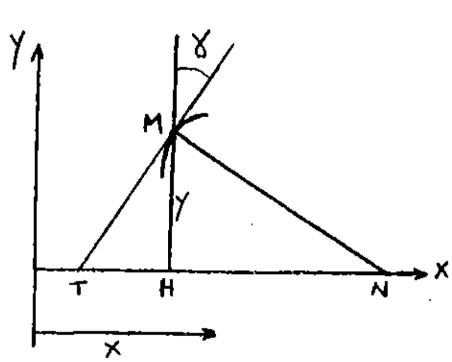
- Equality of arc lengths: $ds_{wheel} = ds_{ground}$

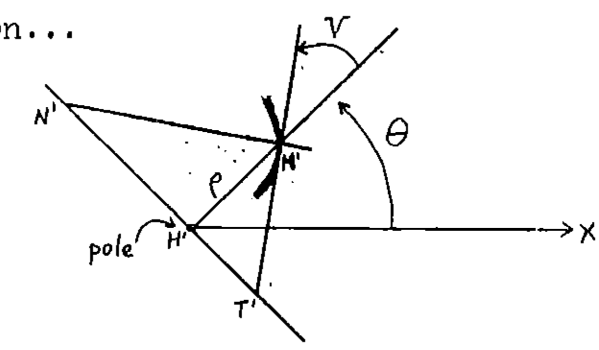
for $d\rho^2 + \rho^2 d\theta^2 = dy^2 + dx^2$

The elementary triangle of Leibnitz (dx, dy, ds) is conserved in the G.T.. So, if we establish a table of correspondence between the constituents of Leibnitz's elementary triangle in polar coordinates and rectangular coordinates, maintaining similar definitions throughout the transformations, the correspondences are exact. Precisely, the equality holds only for the points linked by contact during the movement.

-ground- Rectangular (y,x)	-wheel- Polar (ρ,θ)
point M	point M'
T H	T' H'
M N	M' N'
y	ρ
Angle: γ	Angle: V
H N	H' N'

and so on...

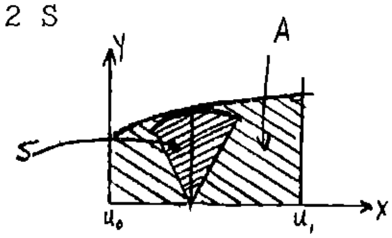




Note: Some authors call Gregory's direct transformation evoluta (G.T.) and involuta (G.T.$^{-1}$).

- Areas: It is possible to prove that the area enclosed within the two extreme ordinates and the ground-curve is twice the area between the two corresponding vector radii and the wheel-curve. For:

$$A = \int_{u_0}^{u} y\,dx = \int_{\theta_0}^{\theta} \rho \cdot \rho\, d\theta = 2\left[\frac{1}{2}\int_{\theta_0}^{\theta} \rho^2\, d\theta\right] = 2S$$



The Steiner-Habicht Theorem

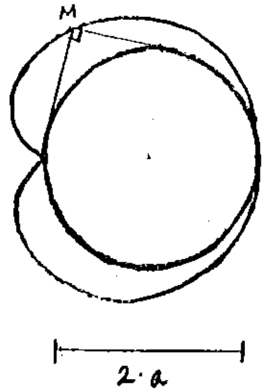
The roulettes and the wheels defined with Gregory's Transformation are associated by the following theorem (Steiner 1846 - Habicht 1882)

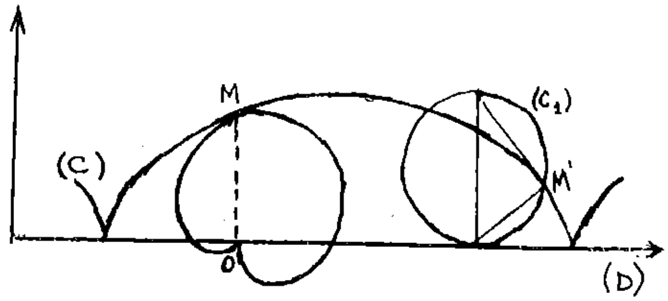
Reminder: The pedal of a curve with respect to the pole is the locus of the pole's orthogonal projection onto the current tangential line.

Theorem : If (c) is the pole's roulette of a curve (C_1), defined in polar coordinates, on a line (D) then the pedal curve of (C_1) with respect to the pole is a wheel corresponding to the ground (C), the hub of which moves on the line (D).

An example will help in the assimilation of the result that is not very difficult to prove:

The roulette of a point on a circle rolling on a line (D) describes a cycloid, a classical result. The circle's pedal with respect to one of its points is a cardioid.





-circle : $\rho = 2 \cdot a \cdot \cos\theta$

its pedal, the cardioid:

$\rho = 2 \cdot a \cdot \cos^2 \left(\frac{\theta}{2}\right)$

-the roulette of (C_1) is a cycloid: $x = a \cdot (t - \sin t)$, $y = a \cdot (1 - \cos t)$

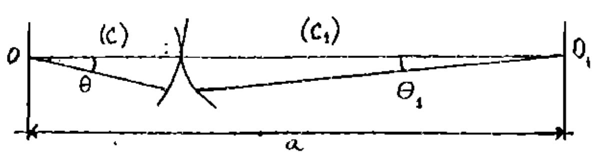
we have: $t = \theta$

According to S & H's theorem, the cardioid rolls on the inner edge of the cycloid and 0 describes the line (D).

If we employ the theorem in the opposite way, we can determine the anti-roulette with the negative pedal (the anti-roulette is that curve which arises when we know the roulette and the line).

Rolling curves and adaptable wheels

Two curves that can roll one on the other, without slipping, and around two fixed poles 0 and 0_1 (at the distance a) in the plane, are called a couple of rolling curves.



If one of the two curves is given in polar coordinates: $\rho = f(\theta)$ then the other one can be simply determined:

We call I the center of elementary rotation (C.E.R.), it is located in the line 00_1.

- On one hand : $\rho + \rho_1 = a$ (3)

- On the other hand, the two curves have a tangent contact at the C.E.R.: I, that gives another condition:

$$\rho \cdot \frac{d\theta}{d\rho} + \rho_1 \cdot \frac{d\theta_1}{d\rho_1} = 0 \qquad (4)$$

This last equation can be reduced to the simpler form:

$\rho \cdot d\theta = \rho_1 \cdot d\theta_1$ (5) because

$d\rho = - d\rho_1$ (By differentiating (3))

We have the equality of lengths for corresponding arcs of (C) and (C_1).

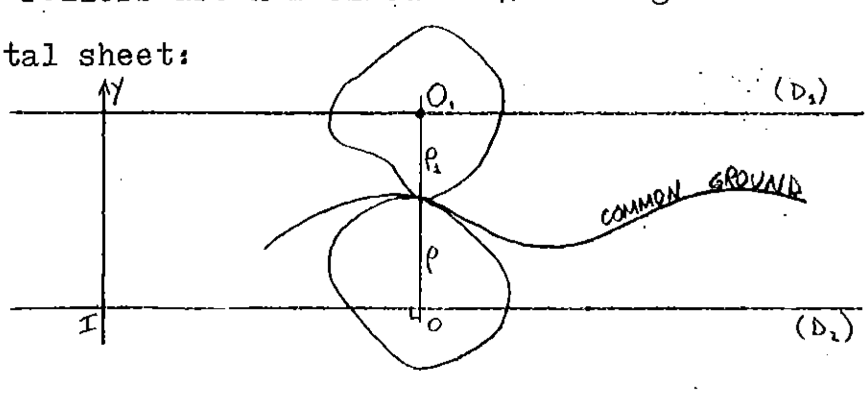
Finally : $\rho_1 = a - \rho$

and $d\theta_1 = \frac{\rho \cdot d\theta}{\rho_1} = \frac{\rho \cdot d\theta}{a - \rho} \longrightarrow \theta_1 = \int_{\theta_0}^{\theta} \frac{\rho \, d\theta}{a - \rho}$; $(\rho \neq a)$

Only one integration is necessary to determine the curve (C_1) when we know (C).

The theory of rolling curves was first investigated by L. Euler (1707-1783) and can be connected to the adaptable wheels in the following way:

Two wheels adapted to a common ground are curves which roll one on the other. The locus of the pole-hub (O, O_1) are two parallel lines (D, D_1) at the distance a which is the distance between the fixed poles of the rolling curves. We get, as it were, a rolling mill in which the rollers are not circular and the ground takes the place of the metal sheet:



If, for example, we fix one of the wheels (O) and roll the ground and the other wheel (O_1) on the first one, then O_1 describes a circle with center O and radius a. This circle is also the envelope of (D_1) linked to (O_1) and (D) passes constantly through O.

We suppose that the parametric ground-equations are $x = x\ (u)$ and $y = y\ (u)$ then: ((D) is the x-axis).

Wheel (O)	Wheel (O_1)
$\rho = y$, $\theta = \int \frac{dx}{y}$ $(y \neq 0)$	$\rho_1 = a-y$, $\theta_1 = \int \frac{dx}{a-y} = \frac{\rho\, d\theta}{a-\rho}$ $(y \neq a)$

These results also give the solution of the following problem: Find the curve (C_2) so that the roulette of a point (O_1) in the plane of (C_1) is a circle.

Let us examine some simple examples in detail:

-A- (1) the ground is the Tchirnhausen's Cubic:

$$\begin{cases} y = a(1 - u^2) \\ x = a(u - \frac{u^3}{3}) \end{cases}$$

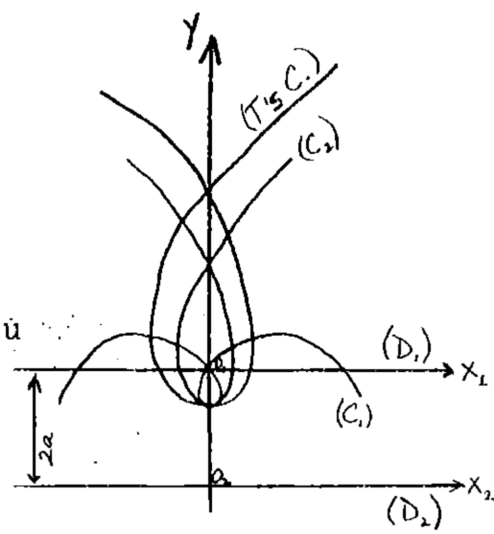
then the wheel corresponding to (D_1) as x-axis is:

$$\rho = y = a(1 - u^2)$$

and

$$\begin{cases} \theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{a(1-u^2)}{a(1-u^2)}\, du = u \\ \rho = a(1 - \theta^2) \longrightarrow \text{Wheel } (C_1) \end{cases}$$

and finally O_1 will move on (D_1).



(2) the ground is the same Tchirnhausen's cubic but the line (D_2) is parallel to (D_1) at the distance 2·a under (D_1).

With this premise, the ground equations are:

$$\begin{cases} y = a(1 + u^2) \\ x = a(u - \frac{u^3}{3}) \end{cases}$$

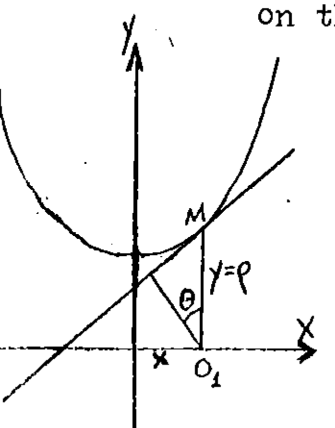
and then the corresponding wheel-equations are:

$$\begin{cases} \rho = y = a(1 - u^2) \\ \theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{a(1-u^2)}{a(1+u^2)}\,du = 2\cdot \text{Arctan}(u) - u \end{cases}$$

if $u = \tan\alpha$ then $\begin{cases} \rho = \dfrac{a}{\cos^2\alpha} \\ \theta = 2\cdot\alpha - \tan\alpha \end{cases} \Longrightarrow$ wheel (C_2)

This last curve and $= a(1 - \theta^2)$ — (C_1) form a couple of rolling curves at the distance 2a between poles. The curve (C_1) is the pedal of (C_2) with respect to the pole (particular property).

-B- The ground is the catenary $y = a\cdot\cosh(\frac{x}{a})$ and the hub moves on the x-axis: $= y = a\cdot\cosh(\frac{x}{a})$



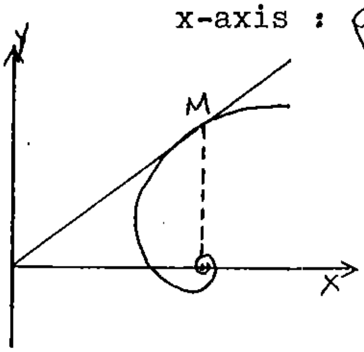
$$\theta = \int_0^x \frac{dx}{y} = \int_0^x \frac{dx}{a\cosh(\frac{x}{a})} = \text{Arctan}(\sinh(\tfrac{x}{a}))$$

then: $\tan\theta = \sinh(\frac{x}{a}) \longrightarrow \cos\theta = \dfrac{1}{\cosh(\frac{x}{a})}$

finally: $= \dfrac{a}{\cos\theta}$, a line.

The line can roll on the catenary in such a way that the locus of the pole is the x-axis.

-C- The ground is a line $y = a\cdot x$ and the hub moves on the x-axis : $\rho = y = a\cdot x$



$$\theta = \int_{x_0}^{x} \frac{dx}{a\cdot x} = \left[\frac{1}{a}\,\mathrm{Log}\left(\frac{x}{x}\right)\right]$$

$$\rho = a\cdot x \qquad a\cdot\theta = \mathrm{Log}\left(\frac{\rho}{\rho_0}\right)$$

then: $\rho = \rho_0 e^{a\theta}$ the wheel is a logarithmic spiral.

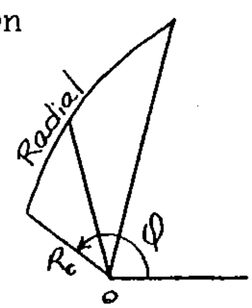
(Other examples are presented at the end of the article.)

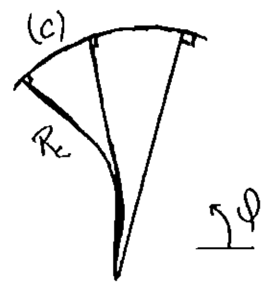
Radials and Mannheim's curves

The Radial of a Curve

If we commence at the origin, 0, in a direction-conserving manner and carry out the radius of curvature of a given curve (C), the locus of the other extremity is the Radial of (C).

$R_c = f(\varphi)$ - polar equation





Mannheim's curve

Ceasaro proposed an intrinsic system of coordinates which uses the radius of curvature: R, and the arc length: s, as measured from an initial point on the curve. In these intrinsic coordinates the form of the equation is: $g(R,s) = 0$

We can interpret this last equation in the rectangular coordinates (A, x, y) without changing the form of the equation $g(y,x) = 0$. That gives a new curve (C') called Mannheim's curve of (C).

It is important to notice :

1) that y corresponds to R and

x corresponds to s in the intrisic equations.

2) changing the reference system must be discerned as a real transformation even if the equation is conservative.

All this leads to the following property:

<u>Theorem</u> : The radial of a curve (C) is that wheel formed by the adaptation of Mannheim's curve of (C) taken as the ground. In other words, together the Mannheim's curve and the Radial of (C) constitute a ground-wheel couple of correspondence.

Here is an example : (C) is the evolute of a circle

$$\left.\begin{aligned} x &= a(\cos\theta - \theta \sin\theta) \\ y &= a(\sin\theta - \theta\cos\theta) \end{aligned}\right\} \Longrightarrow (C)$$

$R = 2\cdot a\cdot s$ is the intrinsic equation.

- The radial of (C) is Archimede's Spiral: $= a\cdot\theta$

- The Mannheim's curve of (C) is the parabola: $y^2 = 2\cdot a\cdot x$

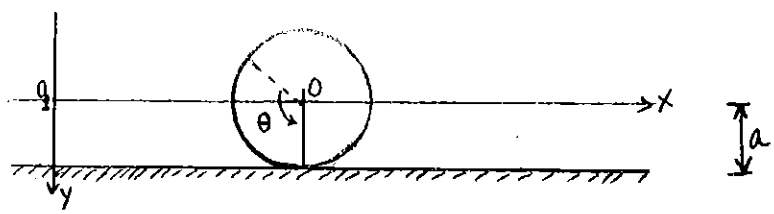
and we again meet the result discovered by mathematicians in the 17th century.

Gregory's transformation also gives the radial of a plane curve when we know the intrinsic equation: $g(R, s) = 0$.

At the end of this quick cursory glance of the adaptable wheels theory and Gregory's transformation we can ask a practical question: what is the use of such an adaptable wheel?

We have to notice that the circular wheel is the simplest of the adaptable wheels:

- the ground is a horizontal line $y = a$
- the wheel is the circle $= a$; $\theta = \frac{x}{a}$



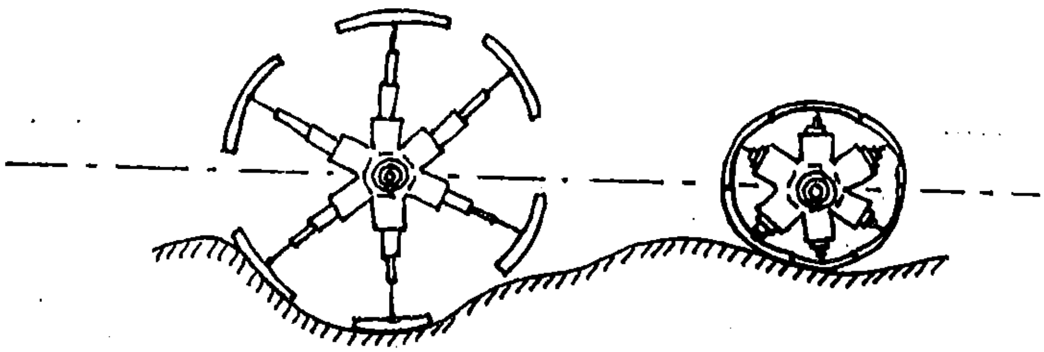
An adaptable wheel could come back to the circular shape when it is necessary, for example, to travel on our roads. However it could take the ski-shape to go down snowy slopes!

Of course, there are some technical problems to resolve before it is possible to produce adaptable wheels, but there is no major difficulty.

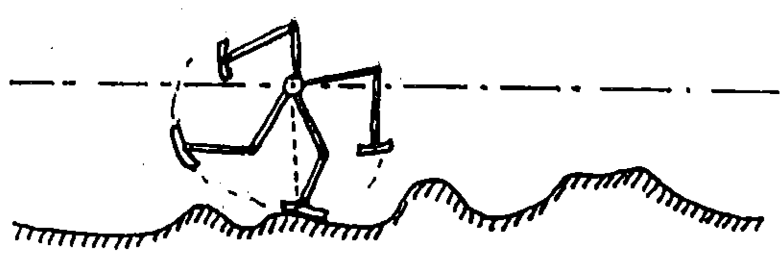
Some mechanical systems could make a compromise between the adaptable wheels and articulated or telescopic limbs (this would allow for the possibility of reversible movement).

A possible configuration may be a hub that would be surrounded by a few telescopic limbs with a flexible foot at the extremity.

This system, technically simpler than the continuous adaptable wheel, is very similar to human legs that have an articulation at the place of the hips. With wheels of this type and an electronical detector able to explore the ground before it, a vehicle could move on a ground of uneven rocks.



- Telescopic limbs -



- Articulated limbs -

Nevertheless the traditional wheel, due to its simplicity and strength, will continue its long career. New solutions would meet applications only in limited and particular fields.

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