CATALAN'S CURVE
from a 1856 paper by E. Catalan
Part - VI

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Abstract

We use theorems on roulettes and Gregory's transformation to study Catalan's curve in the class $C_2(n,p)$ with angle $V = \pi/2 - 2u$ and curves related to the circle and the catenary. Catalan's curve is defined in a 1856 paper of E. Catalan "Note sur la theorie des roulettes" and gives examples of couples of curves wheel and ground.

1 The class $C_2(n,p)$ and Catalan's curve

The curves $C_2(n,p)$ shares many properties and the common angle $V = \pi/2 - 2u$ (see part III). The polar parametric equations are:

$$\rho = \frac{\cos^{2n}(u)}{cos^{n+p}(2u)} \qquad \theta = n\tan(u) + 2.p.u$$

At the end of his 1856 paper Catalan looked for a curve such that rolling on a line the pole describe a circle tangent to the line. So by Steiner-Habich theorem this curve is the anti pedal of the wheel (see part I) associated to the circle when the base-line is the tangent. The equations of the wheel $C_2(1,-1)$ are :

$$\rho = \cos^2(u) \qquad \theta = \tan(u) - 2.u$$

then its anti-pedal (set $p \to p+1$) is $C_2(1,0)$:

$$\rho = \frac{\cos^2 u}{\cos 2u} \qquad \theta = \tan u \qquad \longrightarrow \rho = \frac{1}{1-\theta^2}$$

this is Catalan's curve (figure 1).

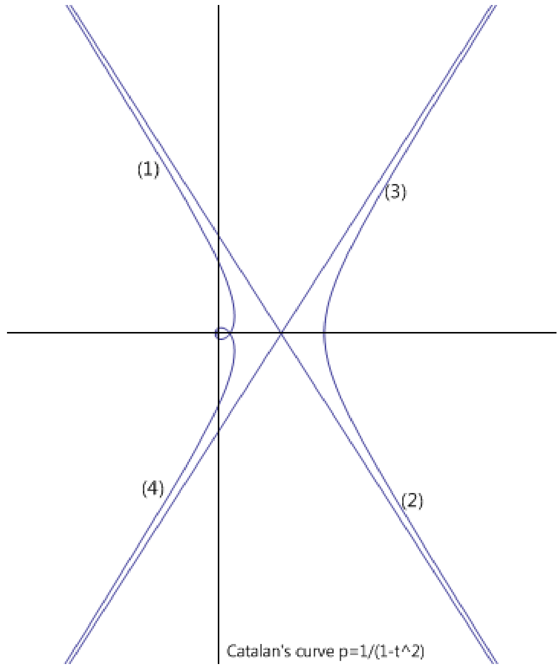

Figure 1: Catalan's curve : $\rho = 1/(1-\theta^2)$

2 Curves such that the arc length $s = \rho.\theta$ with initial condition $\rho = 1$ when $\theta = 0$

This relation implies by derivating :

$$ds = \rho d\theta + d\rho\theta = \sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2} d\theta$$

Squaring and simplifying give :

$$2\rho\theta d\theta \ d\rho + (d\rho)^2\theta^2 = (d\rho)^2$$

So we have the evident solution $d\rho = 0$ this the traditionnal circle : $\rho = 1$ But there is another one more exotic given by the other part of equation after we simplify by $d\rho \neq 0$. We have :

$$\frac{d\rho}{\rho} = \frac{2.\theta.d\theta}{1-\theta^2}$$

that leads to the other solution :

$$\rho = \frac{1}{1-\theta^2}$$

It can verified by direct computation that $s = \frac{\theta}{1-\theta^2}$ We have seen that this curve is related to the circle : when rolling on a line the pole describes a circle tangent to the line and we have $s = \rho.\theta$.

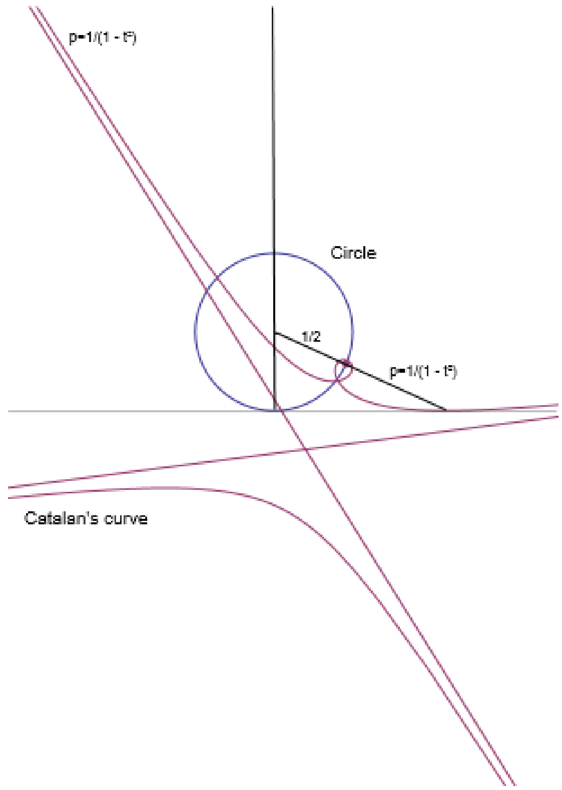

Figure 2: The roulette of courbe $\rho = 1/(1-\theta^2)$ is a circle tangent to the base.

3 The graph of Catalan's curve

We distinguish four parts in the curve. These parts are separated by the axis of symmetry and the two asymptotes in direction $\theta = \pm 1$. The two positive parts (2) - (3) for $-1 < \theta < +1$ when rolling correspond to the upper half of the circle. The two other branches (1) : $-\infty < \theta < -1$ and (4) : $1 < \theta < +\infty$ each have an asymptotic part - the two asymptotes for angles ± 1 passing at $(\rho = 1/(2\sin 1), \theta = 0)$ - and spiral part with an asymptotic pole and they correspond (when rolling on the tangent) to the two symmetric lower quarters of the circle. A parametrisation is $\theta = \tanh(\alpha)$ for the positive branch and $\theta = \coth(\alpha)$ for the two other branches. The tanh and coth are different functions and permit to separate the two parts of Catalan's curve which is double or quadruple if we take in count the axial symmetry.

4 Catalan's curve is the evolute of the spiral tractrix

It can be shown that the evolute of the spiral tractrix is $\rho = \frac{1}{1-\theta^2}$
This spiral is a class $C_1(n,p)$ curve (n=-1, p=1) with parametric polar equations:

$$\rho = \cos(u) \qquad \theta = \tan(u) - u$$

and is also the inverse(set $n \to -n$) of the involute of the circle $C_1(1,-1)$:

$$\rho = \frac{1}{\cos(u)} \qquad \theta = \tan(u) - u$$

The evolute of which is the circle

$$\rho = 1 \qquad \theta = \tan(u)$$

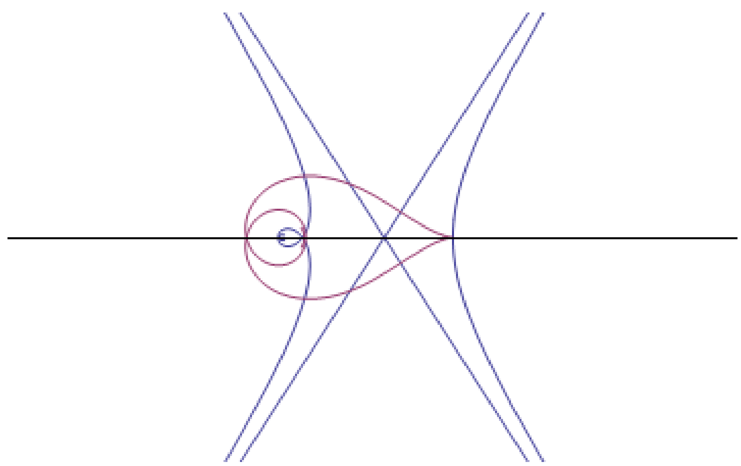

Figure 3: Spiral tractrix and it's evolute : Catalan's curve $\rho = 1/(1-\theta^2)$

5 Catalan's curve, Catenary, hyperbolic spiral wheel and exponential ground

1 - The polar equation of the Catalan's curve can be written in the form :

$$\rho = \frac{1}{1-\theta^2} = \frac{1}{2}.\Big[\frac{1}{1-\theta} + \frac{1}{1+\theta}\Big]$$

is a half sum of two hyperbolic spirals. So Catalan's curve can be seen as a catenary in polar coordinates.
And this is similar to the catenary equation as the half sum of two exponentials :

$$y = \frac{1}{2}.[e^x + e^{-x}]$$

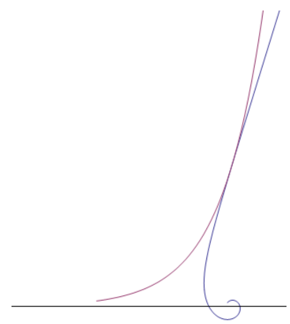

Figure 4: Hyperbolic spiral rolling on exponenttial : $y = e^x$ base line = asymptote

2 - The hyperbolic spiral is a wheel for the exponential curve for the asymptote as base line.

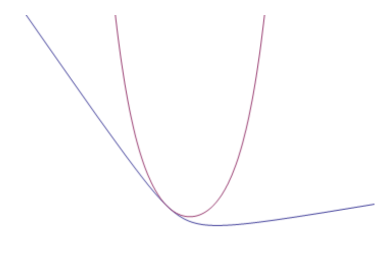

Figure 5: Upper Catenary as the ground, Catalan's curve $\rho = 1/(1-\theta^2) \quad |\theta| < 1$ as the wheel.

3 - The Catalans's curve is the evolute of the spiral tratrix and the catenary is the evolute of the tractrix.

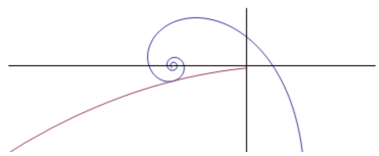

Figure 6: Downward catenary as the ground and Catalan's curve $\rho = \frac{1}{1-\theta^2} \quad |\theta| > 1$ as the wheel.

6 Analogies between the catenary and Catalan's curve

6.1 Caustic by reflection of the exponential for light rays coming from ∞ and parallele to y-axis and x-axis

6.1.1 Caustic for light rays orthogonal to the asymptote

If the light rays are coming from ∞ parallele to y-axis then It can be proved that the catenary $Y = \cosh(X+1)$ is the caustic of the exponential : $y = e^x$. The same catenary is also the caustic of the other exponential $y = e^{(-x-2)}$.

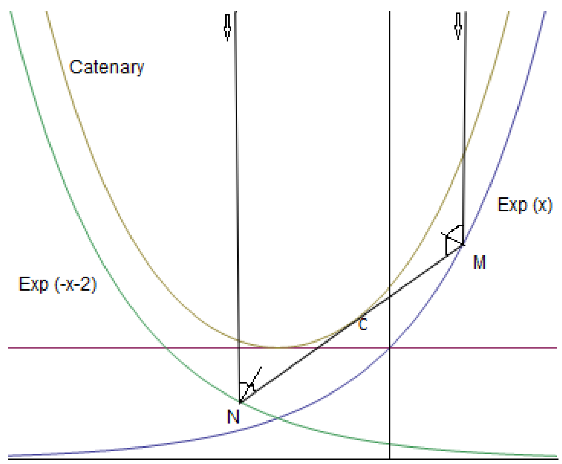

Figure 7: Catenary as the caustic of 2 exponentials.

The caustic by reflection of the Hyperbolic spiral $\rho = \frac{1}{\theta}$ for light rays coming from the pole-origin is the evolute of the homotetic (ratio : 2) of the pedal. This last curve is the Tractrix spiral the evolute of which is Catalan's curve $\rho = \frac{1}{1-\theta^2}$. We have the same chain as for the Exponential, the Tractrix and the Catenary that confirms the geometrical proximity of these two triples of curves. The Tractrix is the anticaustic of the exponential (see Part I : Δ-pedal and anticaustics). So the analogy is formal and geometric.

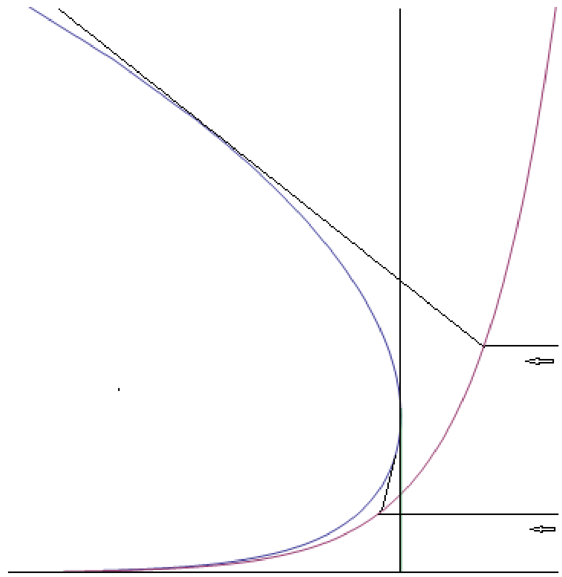

Figure 8: Singular pursuit cuve is the caustic of $y = e^x$ for light rays parallele to asymptote.

6.1.2 Caustic for light rays parallele to the asymptote

If the light rays are parallele to the asymptote x'Ox of $y = e^x$ then the caustic is the singular curve of pursuit when the ratio between the speed of pusued and prosecutor is equal to 1. The parametric equations are :

$$X = \frac{1}{2}.[2x - e^{2x} + 1] \text{ and } Y = 2.e^x$$

The caustics of exponential for the two directions of light rays are illutrated on fig 7 and 8.

7 Orthoptic curves of two exponentials with the same asymptote

We study the set of orthoptic curves of two symmetric exponentials with same asymptote x'Ox, $y_2 = e^{-x-d}$ and $y_1 = e^{x-d}$. d is a kind of the distance between the two curves. Orthoptic associated points on each exponentials are given by $y'_1.y'_2 = -1$. The family of these orthoptic curves parmetrized by the translation d. It is given by the equations :

$$X = x - \tanh(x-d) - \frac{d.e^{(-x+d)}}{\cosh(x-d)} \qquad Y = \frac{1-d}{\cosh(x-d)}$$

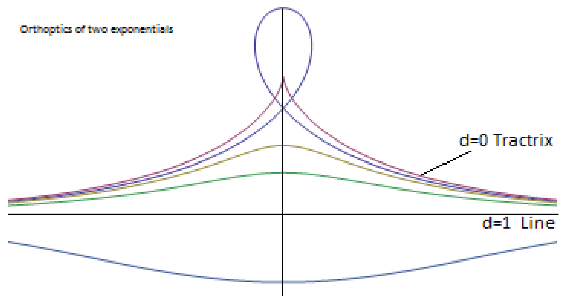

Figure 9: Orthoptic curves of two symmetric exponentials with the same asymptote

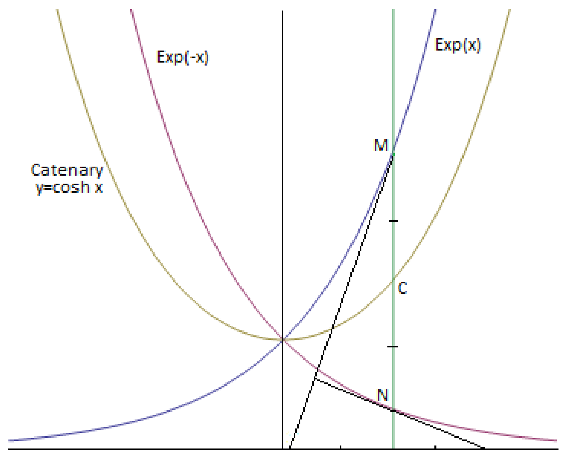

Figure 10: The Catenary, two exponentials (d=0) and 2 orthogonal tangents.

There are two special cases : when $d = 0$ and $d = 1$ which correspond to a generation of the catenary :

1) d=0 is the classical pointwise definition of the Catenary :

$$y = \cosh x = \frac{e^x + e^{-x}}{2}$$

and M is the midpoint of MN, M,N on each of the exponentials on the vertical line at x. The orthoptic curve is the Tractrix, track of the intersection of the two tangents at M and N to the exponentials (see fig.10).
2) d=1 is the (dual) tangential definition of the Catenary as the caustic of the exponential indicated above. The orthoptic curve is the base line x'Ox and the two tangents are orthogonal at I that runs along x-axis (see fig.11).

Gregory's transformation and analogy between Hyperbolic spiral and Exponential explain the fact that adding an angle α to the polar angle θ (around the pole-asymptote O) for the spiral is equivalent to a multiplication by e^d for the

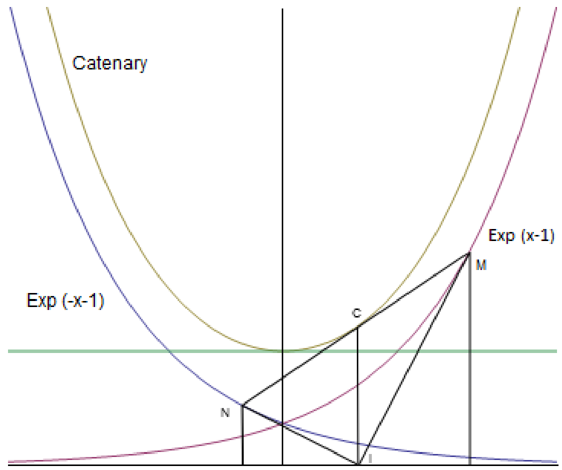

Figure 11: The Catenary, two exponentials (d=1) and 2 orthogonal tangents with subtangents=1. I lies on the line x'x.

exponential or a translation in x along the line-asymptote x'x.

8 The ground corresponding to Catalan's curve as the wheel

The direct Gregory's transform $y = \rho \quad x = \int \rho.d\theta$ applied to Catalan's curve is :

$$x = \int \rho d\theta = \int \frac{d\theta}{1-\theta^2} = \int \frac{1}{2}\Big[\frac{1}{1-\theta} + \frac{1}{1+\theta}\Big] d\theta$$

$$x = \frac{1}{2} \ln \frac{1+\theta}{1-\theta} = \text{Argth}(\theta)$$

We set first $\theta = \tanh(u)$ and second $\varphi = \coth(u) = 1/\theta$ then $x = u$ and :

$$y_1 = \cosh^2 x = \frac{1+\cosh(2.x)}{2} = 1/(1-\theta^2)$$

$$y_2 = -\sinh^2 x = \frac{1-\cosh(2.x)}{2} = \varphi^2/(\varphi^2-1) = 1 - y_1$$

The first curve is an upward catenary with minima is at (0,1), the second is a downward catenary tangent to the base line at the maxima (0,0). So Catalan's curve is a double wheel since it has two parts corresponding to two distinct catenaries as grounds.
These two curves are homothetic in ratio 1/2 of the usual catenary $y = \cosh(x/a)$.

This curve is a double catenary : the ground corresponding to the parts (2) and (3) of Catalan's curve is the positive part of this upward catenary (with $\theta = \tanh(x)$) and the two spirals parts (1) and (4) asymptotic to the pole O

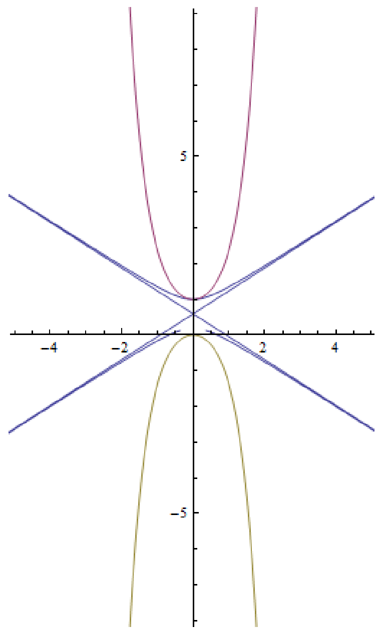

Figure 12: Double catenary as the ground and Catalan's curve $\rho = 1/(1-\theta^2)$ as the wheel.

correspond each to one half of the downard catenary. If a catenary rolls on a line the base of the rolling catenary passes through a point by duality because the line $\rho = 1/\cos\theta$ is a wheel and does not move since a line rolling on a line is immobility. If we fix the wheel then the base line passes throught the pole O which is, in this case, a fixed point. If the Catenary rolls on the same line, the parallele to the base envelope an involute of this point (see part I) which is a circle.

For the circle and its tangent we have similar formulas as above for Double Catenary :
Setting $t = \tan(\theta)$ then $x = \sin 2\theta = 2t/(1+t^2)$ and $\cos 2\theta = (1-t^2)/(1+t^2)$, we have :

$$y_1 = \cos^2\theta = \frac{1+\cos 2\theta}{2} = 1/(1+t^2)$$

$$y_2 = \sin^2\theta = \frac{1-\cos 2\theta}{2} = t^2/(1+t^2) = 1 - y_1$$

9 A particularity of Catalan's curve

The curve has also internal properties that can be read in the equation. The ground made of two catenaries of last section shows that the curve has four

parts of same infinite length (by rolling) :
(1) $-\infty < \theta < -1$
(2) $-1 < \theta \leq 0$
(3) $\quad 0 \leq \theta < +1$
(4) $+1 < \theta < -\infty$
These branches can be associated by a simple pointwise map. Indeed :

$$\rho_{1/\theta} = \theta^2/(\theta^2 - 1) = 1 - \rho_\theta$$

$$\boxed{\rho_{1/\theta} + \rho_\theta = 1}$$

and we see that the two points $P(\theta)$ and $Q(1/\theta)$ are on the same vertical in the couple Catalan/wheel - Double Catenary/ground. We can associate arcs (1) and (2) on one hand then arc (3) and (4) on the other hand so that the moving pole of one part describes by rolling on the other part a circle around the pole of the other fixed in the plane because of the last relation above. And we can exchange the fixed and rolling curves.
The correspondance inside Catalan's curve between points associated by angles as a kind of angular inversion.
$\theta_1.\theta_2 = \pm 1$ then $\rho_1 + \rho_2 = 1$. Associated points are at ∞ for $\theta = \pm 1$ and for the point $\theta = 0 \quad \rho = 1$ it is the origin where $\theta = \pm\infty$.
This is an example of "self-rolling curve". Parts of the same curve are corresponding profiles for rolling about two poles at distance=1.
We know that the element of arc is :

$$ds_\theta = \frac{1 + \theta^2}{(1 - \theta^2)^2} d\theta$$

and the arc element for the corresponding value $1/\theta$ is :

$$ds_{1/\theta} = -\frac{1 + \theta^2}{(1 - \theta^2)^2} d\theta = -ds_\theta$$

There are three base lines in the plane of the double catenary that give only two different equations for the wheel : these are the two tangents at the extremas of the double catenary and the equidistant line between them which gives the wheel $\rho = 1/\cos\theta$ a line in polar coordinates. The two other base lines correspond to Catalan's curve as the wheel. Pieces (1) and (4) associated to the lowest tangent and pieces (2) + (3) to the upward catenary. So if we fix arcs (2) and (3) the two parts (1) and (4) can roll on the fixed arcs and describe a circle about the pole at distance 1 and if we fix (1) and (4) then (2) and (3) can roll on the fixed arcs and describe a circle about the pole at distance 1. Catalan's curve and the line $\rho = 1/\cos\varphi$ are a couple of rolling curves about two fixed points at distance 1/2.
The pole of Catalan's curve (CC) describes :
1 - A line tangent at the top of the Double Catenary if rolling on the double Catenary,

2 - A circle of radius 1/2 if CC rolls on a line,
3 - A circle of radius 1 if CC rolls on the other part of CC : (1) rolling on (2) and (4) rolling on (3).

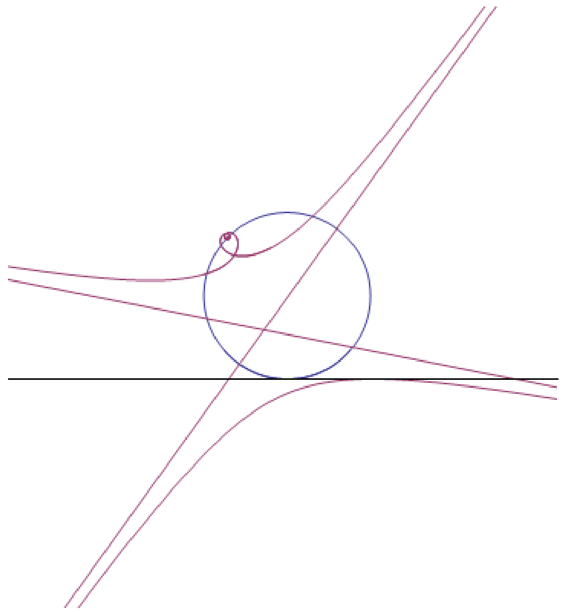

Figure 13: Circle is the roulette on a line of Catalan's curve $\rho = 1/(1-\theta^2)$

10 Weierstrass substitution and parametrizations of the circle and of Catalan's curve

A parametrisation of the circle is:

$$x = \cos\theta \qquad y = \sin\theta$$

and an algebraic parametrization with the tangent of half-angle $t = \tan\frac{\theta}{2}$ is :

$$x = \frac{1-t^2}{1+t^2} \qquad y = \frac{2.t}{1+t^2}$$

A method in calculus - Weierstrass substitution - for integrating rational expressions involving trigonometric functions (sinus, cosinus and tangent) is the use of a new variable $t = \tan(\theta/2)$. The formulas are :

$$\sin(\theta) = 2.t/(1+t^2) \qquad \cos(\theta) = (1-t^2)/(1+t^2)$$

$$\tan(\theta) = 2.t/(1-t^2) \qquad \text{and } d\theta = 2.dt/(1+t^2)$$

The element of arc length of the circle is :

$$ds = \frac{2.dt}{1+t^2} \to s = 2.\arctan[\tan(\theta/2)] = \theta$$

Catalan's curve is $\rho = 1/(1-\varphi^2)$ and if we use this time the hyperbolic tangent of half-angle ($\varphi = \tanh\frac{x}{2}$ or $\varphi = \coth\frac{x}{2}$) as the parameter then :

$$\sinh(x) = 2.\varphi/(1-\varphi^2) \qquad \cosh(x) = (1+\varphi^2)/(1-\varphi^2)$$
$$\tanh(x) = 2.\varphi/(1+\varphi^2) \qquad \text{and } dx = 2.d\varphi/(1-\varphi^2)$$

$$dx = \frac{2.d\varphi}{1-\varphi^2} \to x = 2.\arg\tanh[\tanh(x/2)]$$

These computations confirm that the dual element of the angle θ is the length x since a rotation for wheel corresponds to a translation along the x axis in the plane for the ground.

This article is the 6^{th} part on a total of 10 papers about Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english version.

References :

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F. Gomez Teixeira - Traite des courbes speciales remarquables Chelsea New York 1971 (3 tomes)
Nouvelles Annales de Mathematiques (1842-1927) NAM - Archives Gallica
Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica