Caustics by reflection
Curves of direction
Rational arc length
Part - XII

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Abstract

Caustics by reflection and curves of direction have, when algebraic as shown by Laguerre and Humbert, a rational arc length. Examples of low order caustics are drawn with help of geometric definition of catacaustics. For conics the caustic is in general of order 6. We present links with classes of curves of direction studied by Cesaro, Balitrand and Goormaghtigh in NAM between 1885 and 1920.

1 Plane caustics by reflection.

Caustics of plane curves are defined as the envelope of the light rays coming from a ponctual source and reflected or refracted in a given curve. We only study in this paper the first type of caustics (reflection) and never caustics by refraction so we drop the precision and call them caustics.
Caustics by reflection (sometimes called catacaustics) seem to go back to Tschirnhausen and his paper of 1682 but Apollonius knew some properties of foci of conics (the light rays from a focus converge in the other focus). For the parbola : parallele rays to the axis of symetry from infinity converge at the only focus, which is a caustic.
Two cases of caustics can be distinguished :
- Light rays coming from a ponctual source of light at finite distance,
- Light rays coming from a point at infinity,
In the first case light rays diverge from a point and in the second light rays are parallele.
The transformation of the initial curve is a tangential transformation that maps tangents of the first curve to tangents of the caustic. The two envelopes are associated curves.

Geometers have found simple constructions to draw the curves from the tangents of the given curve. The light point is a special point of the caustic since its position with respect to the initial curve defines the form of the resulting caustic. An orthogonal trajectory for the reflected rays is called the catacaustic, the evolute of this catacaustic is the searched caustic. It is well known that for ponctual light source a catacaustic of a given plane is the dilated (k=2) pedal of the given reflecting curve. The evolute of the dilated pedal is the caustic.

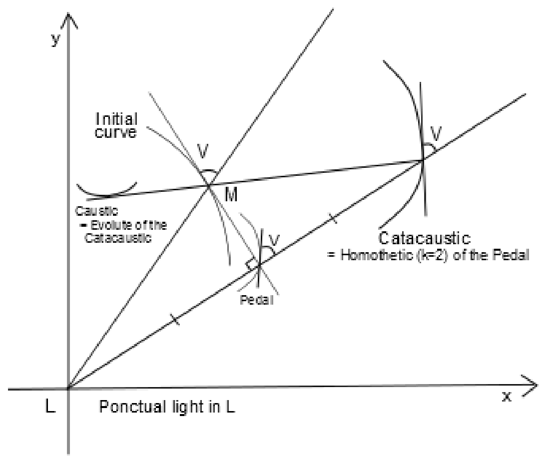

Figure 1: Central caustic of a curve

For a source of light at infinity in y-axis direction the construction is a little different : project the current point of the reflecting curve at H on the x-axis orthogonal to the direction of the light rays. The symetric of H wrt to the tangent at the current point describes the catacaustic the evolute of which is the caustic.

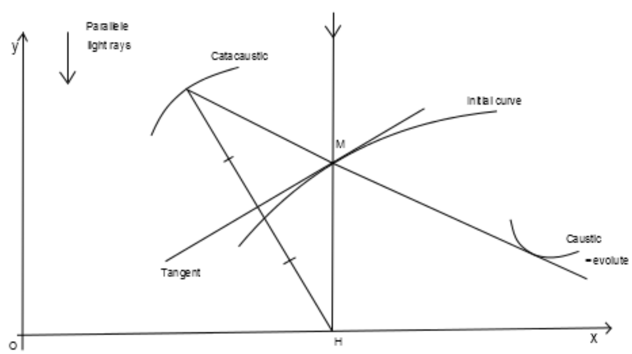

Figure 2: Caustic of a curve for parallele rays to y-axis

2 Caustics by reflection of conics : light point at finite distance.

The non-degenerate conics are the first class of simple curves in the plane : algebraic curves of order 2 and class 2 when non degenerate. It has been proved that in general case the caustics of conics are curves of order 6. Among the examples quoted by G. Humbert are the Nehphroid, the caustic of a circle for parallle rays, the astroid, caustic of a deltoid for parallle rays. The formula for computing the maximum order of a caustic of an algebraic curve of order n in general case - ie without special singularities or peculiarities that could decrease the order - is :

$$\text{Order of the caustic} = 3.n.(n-1)$$

For conics n=2 the caustics are of order 6, for cubics n = 3 the order is 18 and for n=4 then the caustic has order 36. So caustics of algebric curves become complicated special algebraic curves. In fact G. Humbert has given some classic examples and the present paper tries to add new examples of plane caustics by reflection. Caustics of conics are in general of order 6, but there are exceptions, the catacaustic of the parabola which is tangent to the line at infinity, is of order 4 and class 3 - the class is the number of real or complex tangents to the curve that can be drawn from a general point in the plane -. Class and order of conics are in general 2.

The foci of conics are also the foci of catacaustics and of their evolutes the caustics. That is a general property for catacaustics and caustics of algebraic curves. Foci are the point circles of null radius bi-tangent to the plane curve or equivalently the points from which we can lead two tangents through the two circular points at ∞ I, J described by the homogeneous coordinates (1, i, 0) and (1, -i, 0) to the algebraic curve.

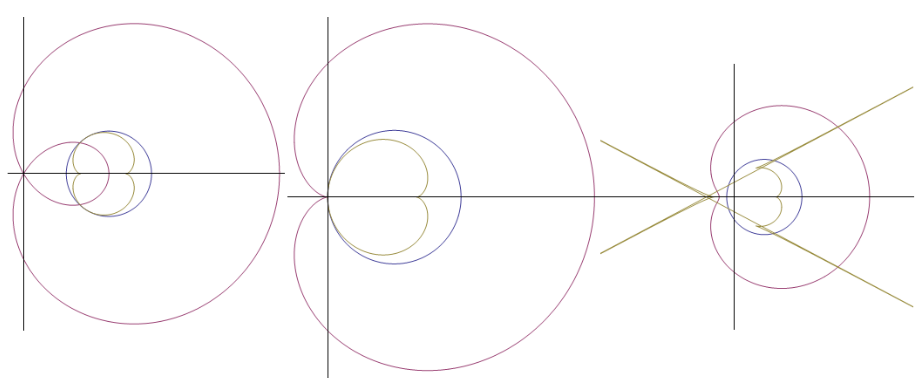

Figure 3: Caustic of the Circle : ponctual light at origin

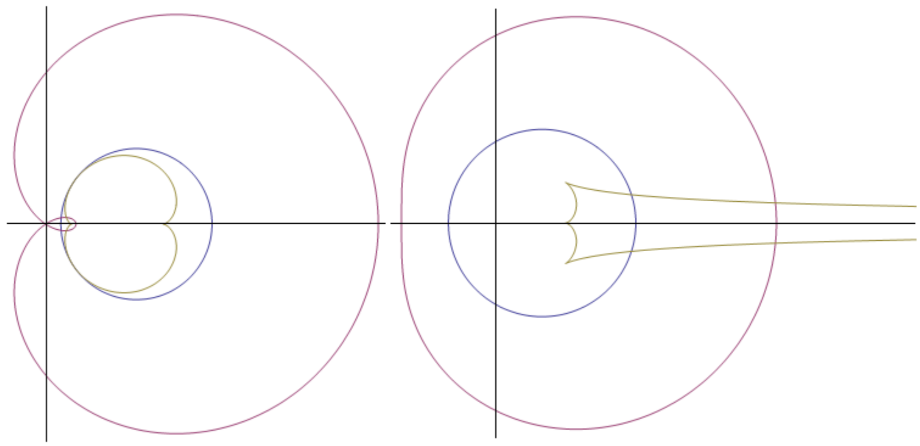

Figure 4: Caustic of the Circle : ponctual light at origin

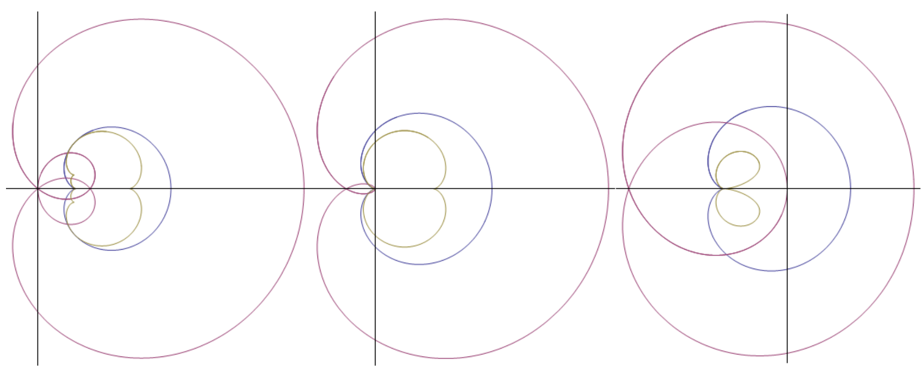

Figure 5: Caustic of the Cardioid : ponctual light at origin

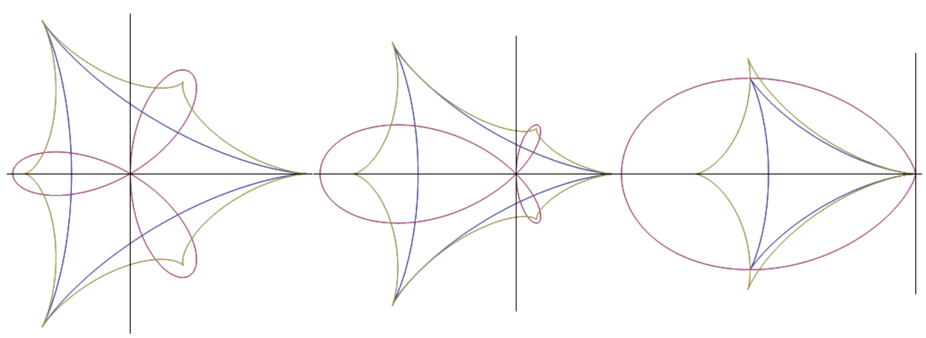

Figure 6: Caustic of the Deltoid : ponctual light at origin

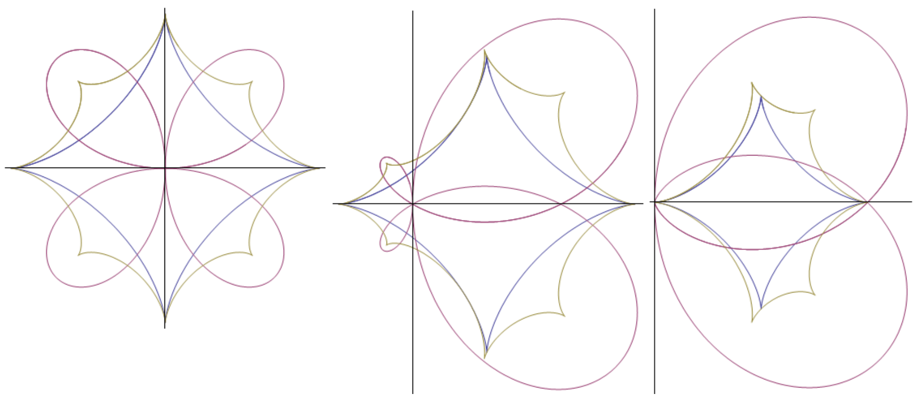

Figure 7: Caustic of the Astroid : ponctual light at origin

2.1 Some examples of caustic for light rays coming from a point L (central caustics) at finite distance.

These are just listed here and can be verified using the above construction of the caustic (CR) as evolute of the pedal (wrt to L) of the initial curve (C).

1 - (C) is a circle, L is a point on (C), (CR) is a cardioid.
2 - (C) is a cardioid $\rho = \cos^2(\theta/2)$, L is the cusp, (CR) is a Nephroid since the pedal of the cardioid is Cayley's sextic $\rho = \cos^3(\theta/3)$ and its evolute is the nephroid.
3 - (C) is a circle, L is any point not on the circle, (CR) is an evolute of a Pascal snail.
4 - (C) is a conic, L is any point not on the conic, (CR) is a curve of order 6, the evolute of a bicircular quartic.
5 - (C) is a generalised sinusoidal spiral $C_1(n, p)$:

$$\rho = \tan^n u \sin^p u, \qquad \theta = n \tan u + pu$$

L is the pole, (CR) is an evolute of the pedal given by $p \to p{+}1$ in the parametric equations of (C).
6 - (C) is an hyperbolic spiral (n=1, p=0) : $\rho = 1/\theta$, L is the pole, (CR) is the tractrix spiral (n=1, p=1) so the evolute is curve of Catalan (see Part VI) : $\rho = \frac{1}{1-\theta^2}$ (see part VI).
7 - (C) is a generalised sinusoidal spiral $C_{2*}(n, p)$:

$$\rho = [\frac{1}{1 - \tan^2 u}]^n \cos^{-p} 2u, \qquad \theta = n \tan u - 2p.u$$

L is the pole, (CR) is an evolute of the pedal given by $p \to p{+}1$ in the parametric equations of (C).
8 - (C) is the curve of Catalan (n=1, p=0) : $\rho = \frac{1}{1-\theta^2}$, L is the pole, (CR) is the evolute of the curve (n=1, p=1) : $\rho = \cos^2 u, \theta = \tan u - 2u$ - see Part III-.

Finally we have equation of the central caustic : $\rho = \frac{\cos u}{3-\tan^2 u}, \theta = \tan u - u$.
9 - (C) is a generalised sinusoidal spiral $C_3(n,p)$:

$$\rho = [\frac{\cos u}{3-\tan^2 u}]^n \sin^p 3u, \qquad \theta = n \tan u - (3p+n)u$$

L is the pole, (CR) is an evolute of the pedal given by $p \to p+1$ in the parametric equations of (C).
10 - (C) is The curve (n=1, p=0) : $\rho = \frac{\cos u}{3-\tan^2 u}, \theta = \tan u - u$, L is the pole, (CR) is the evolute of the curve (n=1, p=1) so the evolute of the following pedal curve : $\rho = \cos^2 u. \sin 2u, \theta = \tan u - 4u$. I have no central parametric equations $\rho(u), \theta(u)$ for the evolute of this pedal.

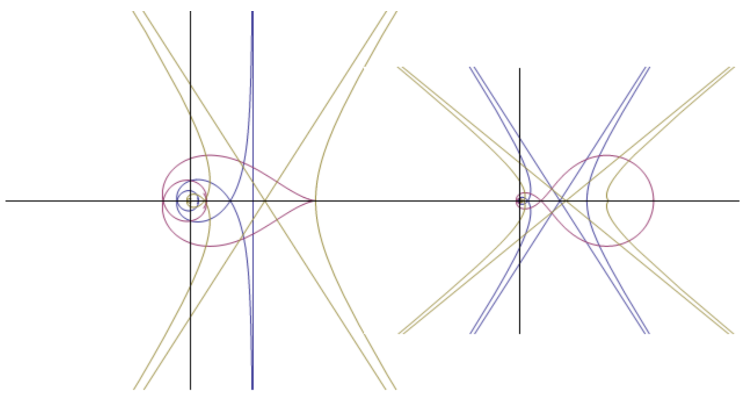

Figure 8: Central caustics of $C_k(n,p)$: ponctual light at origin

Some pictures shows different cases of conics (red), catacaustic (blue) and caustic (beige).

Catacaustics and their evolutes, the caustics of ellipses for central or paralle

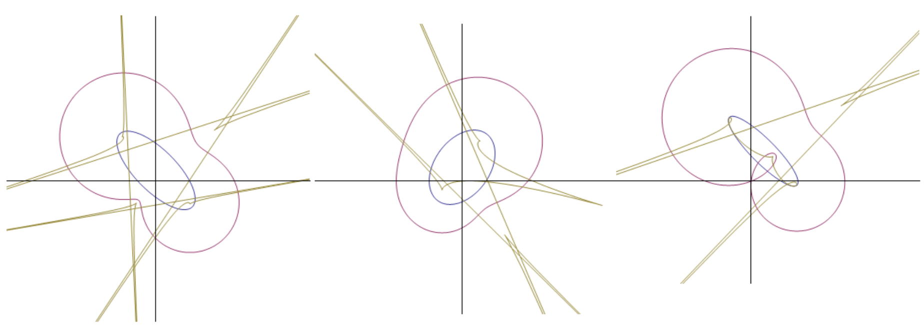

Figure 9: Central caustics of ellipses

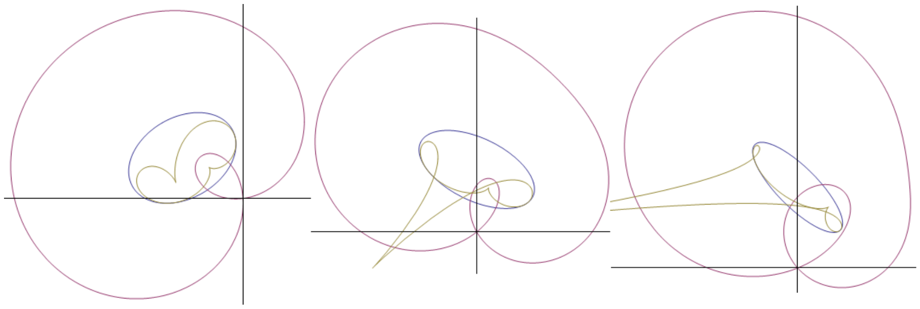

Figure 10: Central caustic of circle and ellipses

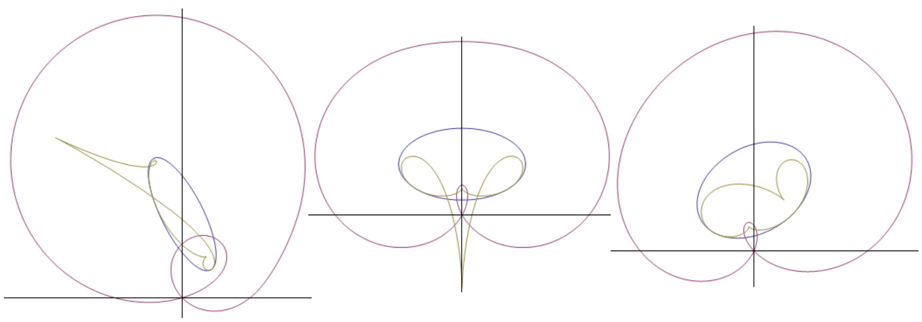

Figure 11: Central caustics of ellipses

light rays present in general two loops or two cusps linked by two arcs and no points at infinity and no inflexion point. And since all involutes or evolutes of this type of curve are of the same kind with two loops or two cups. It gives a stability of forms for these kind of curves by transformations involute/evolute.

The end of this paper will be exclusively be concerned by cautics for parallele light rays and the very closed relation to the important class of curves of direction (CD) studied by E. Laguerre, G. Humbert and others in the Nouvelles Annales de Mathematiques at the end of nineteenth century.

3 Caustics by reflection of conics : light coming from ∞ parallele to y-axis.

3.1 A special case : the parabolic mirror.

In the case of the parabola it is well known since the greeks that the caustic of the parabola for light rays parallele to the axis of the parabola is a point : the focus F of this parabola. So a point can be assimilated to a caustic.

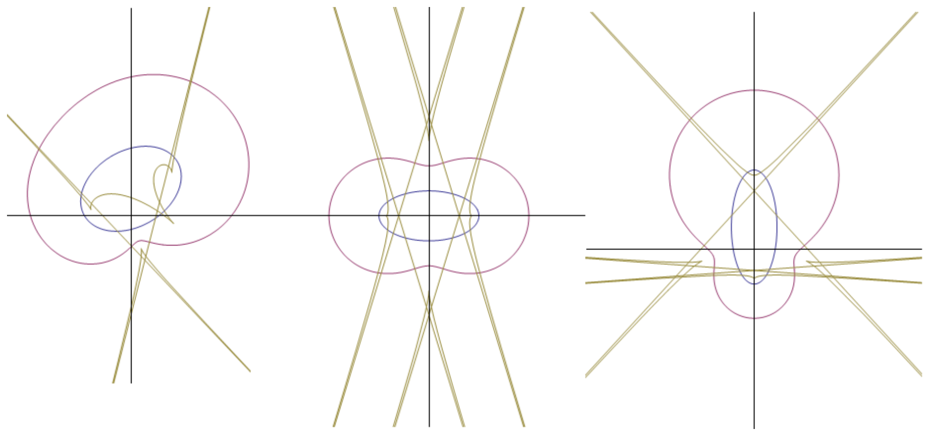

Figure 12: Central caustics of ellipses

3.2 Some examples of caustic for light rays coming from ∞ parallele to y-axis.

These are just listed here and can be verified using the above construction of the caustic (CR) as evolute of the catacaustic of the initial curve (C).

1 - (C) is a parabola ($y = 1 + x^2/4$) (CR) is the focus of the parabola.
2 - (C) is a cycloid with base x-axis, (CR) is a cycloid twice smaller.
3 - (C) is a circle, (CR) is a nephroid.
4 - (C) is a parabola in any position except axis // to y-axis, (CR) is a Tschirnhausen's cubic.
5 - (C) is a deltoid in any position, (CR) is an astroid. (see animation Astroid on http://www.mathcurve.com/)

These three last cases are curious. If for the circle it is trivial since this curve has a rotation symetry, it is not evident for the parabola and the deltoid. Are there other similar cases ? Among cycloidals this particuliarity could rather be often met.

For the catenary $y = \cosh x$:
Parametric equations of the catacaustic (Poleni's curve) :

$$x = 2/\cosh x \qquad y = x - 2\tanh x$$

Parametric equations of the caustic (evolute of Poleni's curve) :

$$x = x - \tanh x \qquad y = \frac{1}{2}[\frac{2 + \cosh^2 x}{\cosh x}]$$

For the tractrix :

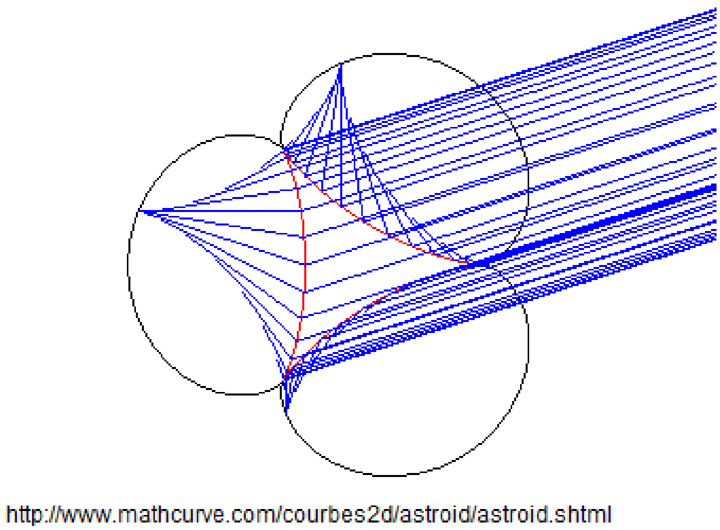

Figure 13: Caustic of the Deltoid : an Astroid

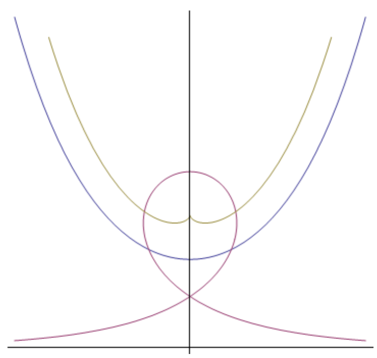

Figure 14: Caustic of the Catenary

$$x = u - \tanh u \qquad y = 1/\cosh u$$

Parametric equations of the catacaustic :

$$x = u + \tanh u - 2\tanh^3 u \qquad y = 2[\frac{\sinh^2 u}{\cosh^3 u}]$$

Parametric equations of the caustic (evolute of the previous curve):

$$x = u - \tanh u + \tanh^3 u \qquad y = \frac{1}{2}[\cosh u - \frac{1}{\cosh u} + \frac{2}{\cosh^3 u}]$$

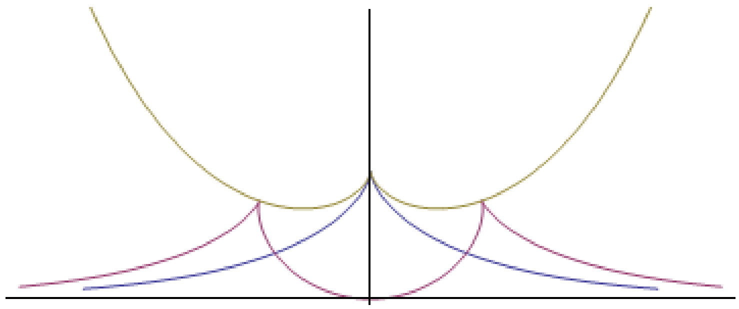

Figure 15: Caustic of the Tractrix

3.3 Caustic of the parabola in general position.

When light rays come from any direction in the plane exept the one parallele to the axis of the parbola then the caustic is the Tschirnhausen's cubic (TC) - see part IV -.
The catacaustics that appear in the above construction are evolutes of the TC and are the curves called cubic hypercycles by Laguerre.
If the Tschirnhausen's cubic is given by equations : $x = 3.t^2$ and $y = 3.t - t^3$ its arc length is $s = 3t + t^3$ then the parametric equations of the cubic hypercycles or parabola catacaustics are :

$$x_1 = \frac{4t^3 + b(1 - t^2)}{1 + t^2}$$

$$y_1 = \frac{t^2(t^2 - 3) + 2bt}{1 + t^2}$$

Its element of arc length is a rational espression (as an involute of a CD) :

$$ds_1 = 2.\frac{b - t(3 + t^2)}{1 + t^2}.dt$$

3.4 Caustic of the ellipse $x^2/a^2 + y^2/b^2 = 1$ for light rays // to y axis.

For an ellipse $x = a\cos t$ and $y = b.\sin t$ centered at O and axis Ox an Oy the catacaustic is a symetric curve and is given by the following parametric equations (with help of a software) :

$$x_1 = a\cos t + 2ab^2 \sin t^2 \cos t/(a^2 \sin t^2 + b^2 \cos t^2)$$

$$y_1 = 2a^2 b \sin t^3/(a^2 \sin t^2 + b^2 \cos t^2)$$

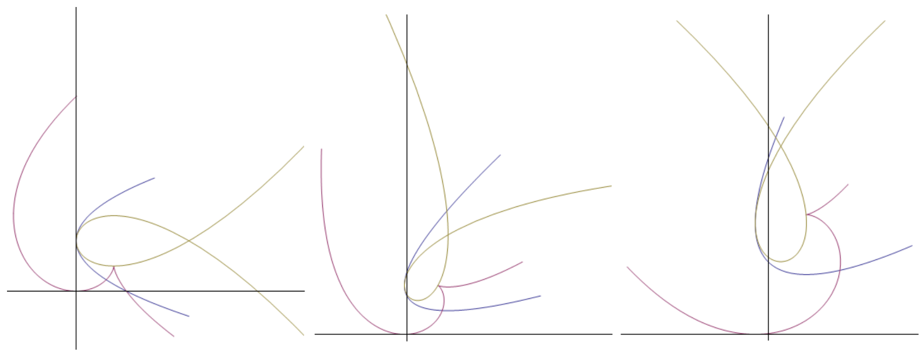

Figure 16: Catacaustics of parabola

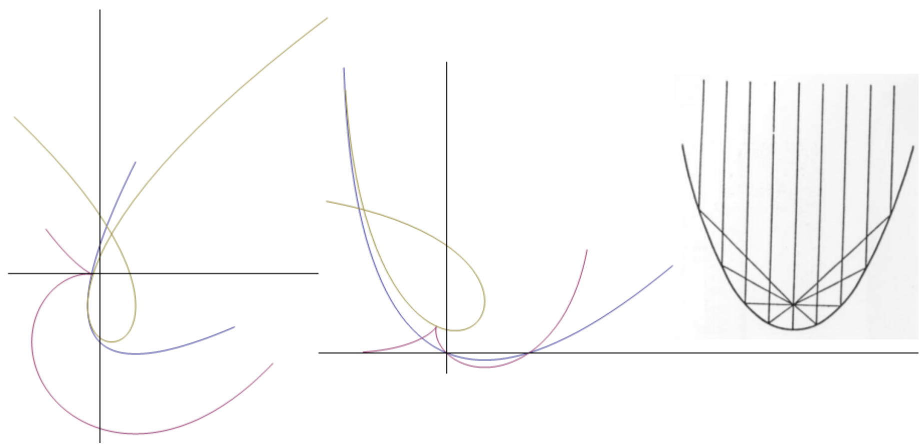

Figure 17: Catacaustics of parabola

And the caustic is the evolute of the catacaustic, its parametric equations are :

$$x_2 = (a\cos t(2a^2\sin t^2 + b^2(2\cos t^2 + (4 - a^2 - 5b^2 + (a-b)(a+b)\cos 2t)\sin t^2)))...$$

$$.../(2(b^2\cos t^2 + a^2\sin t^2))$$

$$y_2 = -(b(-3a^4 + 11b^4 - 8a^2(-2+b^2) + 4(a^4 + 3b^4 + 2a^2(-2+b^2))\cos 2t + (-a^4 + b^4)\cos 4t)\sin t)...$$

$$.../(16(b^2\cos[t]^2 + a^2\sin t^2))$$

(to be verified) The arc length of the catacaustic is given by a rational formula in $\sin t$ and $\cos t$ so in x and y :

$$ds_1 = \frac{a(3.b^2 + (a^2 - b^2)\sin t^2).\sin t}{b^2\cos t^2 + a^2\sin t^2}.dt$$

3.5 Caustic of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ for light rays // to y axis.

For a hyperbola and equations : $x = a.\cosh t$ and $y = b.\sinh t$ there are corresponding formulas with the following expression for the rational arc length of the catacaustic in this special case :

$$ds_1 = \frac{a(-3b^2 - (a^2 + b^2)\sinh t^2).\sinh t}{b^2 \cosh t^2 + a^2 \sinh t^2}.dt$$

(to be verified)
Some cases of caustics for inclined parabola, ellipse and hyperbola are shown. In each picture the three curves are traced : the conic, the catacaustic and its evolute the caustic. The foci of the conics are also the foci of the catacaustic and of the caustic. This property is true for algebraic plane curves in general.
Since the caustics of cubic are in the general case of order 18 it seems out of range to study these curves even with help of software so we will not go futher in this too complicated path.

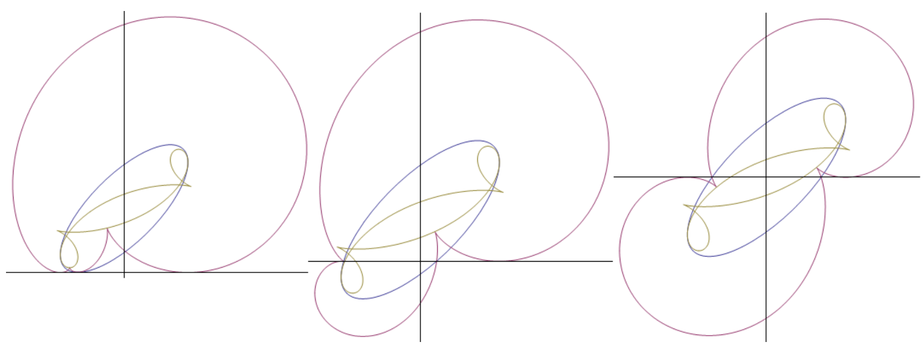

Figure 18: // rays of light : caustic of ellipses

4 Curves of direction (Laguerre).

These curves have special properties. They are rational plane curves with an arc length expressed by a rational formula. A point can be assimilated to CD of class one.
The curves of direction of order 2 and class 2 are the evolutes of the point : the circles.
The next one of order 3 and class 4 is the Tschirnhausen's cubic (TC) and has a rational arc length.
The cubic hypercycles are CD of class 3 and order 3 and are evolutes of the (TC). The general hypercycles are CD of order 6 and class 4. Examples are the Nephroid, the astroid, the oblique astroid. We recall the one of an harmonic

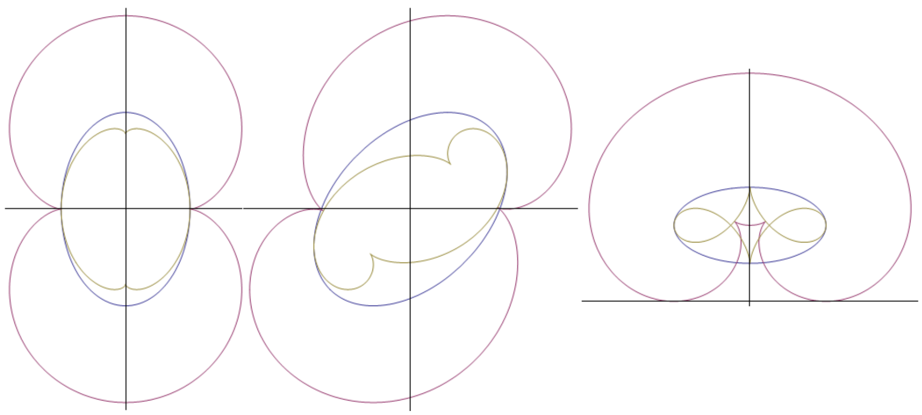

Figure 19: // rays of light : caustic of ellipses

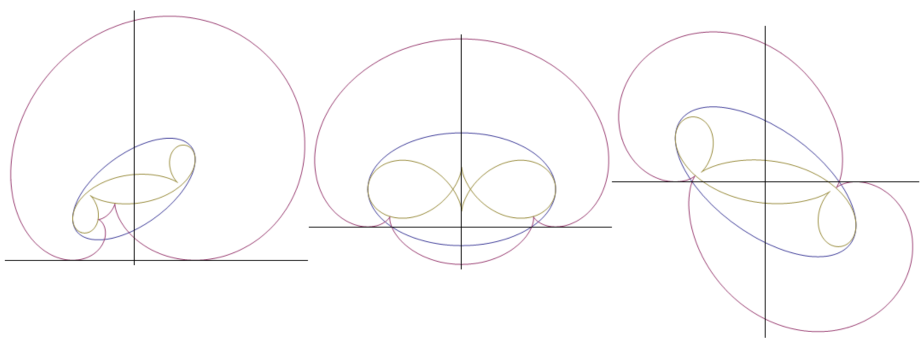

Figure 20: // rays of light : caustics of ellipses

system : two couples of semi-lines (with direction) (A, A') and (B, B') form an harmonic system if they are tangent to a same cycle and if contact points divise harmonically the cycle; A' is the harmonic conjugate of A wrt the couple of semi-lines (B, B').
The definition of hypercycles is given by Laguerre in a paper "on the hypercycles (1882)". The hypercycle is defined by the following property : harmonic conjugates of a semi-line in the plane wrt couples of conjugate tangents have for envelope a circle K. Laguerre presents two interessant classes : the cubic hypercycle (class 3) and the proper hypercycles (class 4). The cubic hypercycle is the only curve of direction of class 3, circular quartic of class 3 tangent to the line at infinity. Cubic hypercycles are the catacaustics of the parabola for parallele light rays.

Fondamental properties of the curves of direction (CD) are the followings :
- If the evolute of a curve is a CD then the curve is a CD (G. Humbert).
- In general the involute of a CD is a CD and a parallele curves of a CD is a

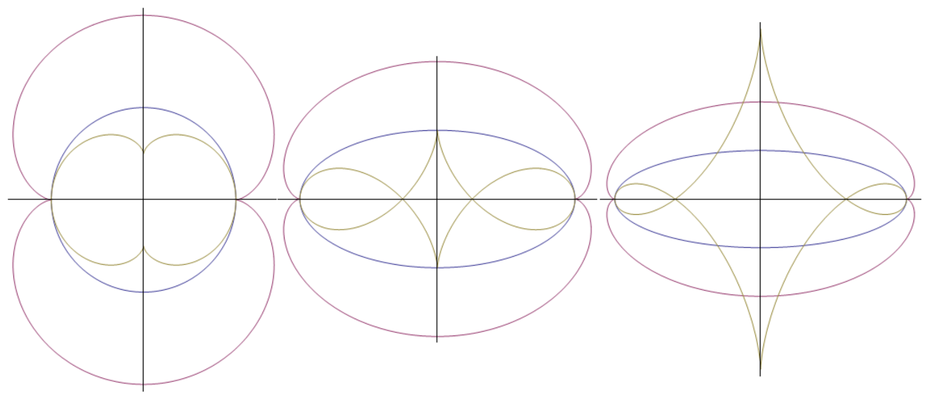

Figure 21: Parallele rays caustics of circle and ellipses

CD (G. Humbert).
Any curve of direction is a catacaustic of an algebraic curve for parallele incident rays and in the other way any catacaustic of an algebraic curve is a curve of direction (E.Laguerre).

A tangential equation in (u,v) is a relation that defines the curve as the envelope of a line $u.x + v.y = 1$. The general equation in tangential coordinates (u, v) for curves of direction is :

$$(u^2 + v^2).F^2(u, v) - \Phi^2(u, v) = 0$$

where F and Φ are integer polynomials in (u,v). Hypercycles are of class 4 or 3 and have the following tangential equation :

$$(u^2 + v^2).(\alpha.u + \beta.v + \gamma)^2 - (A.u^2 + 2Bu.v + C.v^2 + 2D.u + 2E.v)^2 = 0$$

The tangential equation of the Tschirnhausen's cubic is :

$$w^2(u^2 + w^2) = [p(u^2 + v^2 + uv)]^2$$

Humbert has given some infinite classes of curves of direction :
Sinusoidal spiral ($\rho^n = \cos n.\theta$) with $n = p/q$:

$$\rho = \cos^{p/q}\left[\left(\frac{q}{p}\right).\theta\right] \qquad \text{for odd p, q} \in \mathbb{N} \quad p \cap q = 1$$

5 Cesaro curves

These curves can be defined (R is a real length) by :

$$[\rho^2 - R^2]^{(n+1)} = a^{2n}.\rho^2.\sin^2 V$$

the polar equation :

$$d\theta = \frac{(\rho^2 - R^2)^{(n+1)/2}}{\rho.\sqrt{a^{2n}.\rho^2 - (\rho^2 - R^2)^{n+1}}} d\rho$$

and the equation for the radius of curvature:

$$R_{curv.} = \frac{a^n}{(n+1).\sqrt{(\rho^2 - R^2)^{n-1}}}$$

Two known subclasses of these curves correspond to sinusoidal spirals (R=0) and Ribaucour curves ($R = \infty$). All intermediate cases are associated to $0 < R < \infty$ and include cycloidals as special cases.
The radius of curvature can only be infinite or null on the circle $\rho = R$ which can be cut only orthogonally by the Cesaro's curves so inflexions and cusps are on the circle $\rho = R$.

Cesaro curves are functions of two parameter λ, μ and have been studied in his book on intrinsic plane geometry and in NAM by Balitrand, Turriere and others. In (4) Cesaro gave some properties of his curves. And in (8) Goormaghtigh published a synthesis paper that lists many caracteristics of these curves, generalising Ribaucour curves and sinusoidal spirals. But he used parameters different from those used by Cesaro in (4).
The intrinsic equation of Cesaro curves is :

$$s = \int \frac{\lambda}{\sqrt{(\rho/a)^{\mu} - 1}} d\rho$$

In this equation ρ is the current radius of curvature, s is the arc length from a given origin and the parameters λ and μ are constants. If μ= -2 we have cycloidals. If μ= -1 we have parallele curves to the cycloidal with same λ. Cesaro mentions also the alysoides $\mu = 1$, the catenary if $\lambda = 1$, the catenary of uniform strength ($\mu = 4, \lambda = \pm 3$), the lemniscate de Bernoulli (($\mu = 2/3, \lambda = \pm 1/3$), the sinusoidal spirals ($\pm\lambda = \mu - 1$) and Ribaucour curves ($\pm\lambda = (1/2)\mu \neq 1$), etc. Cesaro in (4) also defines 3 sub-classes of his curves by a dilation, in a constant ratio from the current contact point, of the osculator circle.
He distinguishes 3 cases :
-1- Dilated osculator circle passes through a fixed point $\longrightarrow$ Sinusoidal spirals.
-2- Dilated osculator circle is normal to a fixed line : $\longrightarrow$ Ribaucour curves.
-3- Dilated osculator circle is tangent to a fixed line : these when algebraic are curves of direction presented in the two sections below.
The above Cesaro intrinsic equation is a mine of special curves with interesting properties and it is possible to find solutions (by elementary functions) for small integer values of λ and μ.

6 Grounds for curves $C_{2*}(n,p)$ are catacaustics of Ribaucour curves.

The Ribaucour curves are defined as curves such that the radius of cuvature is cut by x-axis in a constant ratio k. We have seen (Parts I and III) that sinusoidal spirals are wheels for Ribaucour curves. If the sinusoidal spiral wheel rolls on the symetric side wrt the current tangent the pole describes a catacaustic of the Ribaucour curve and this catacaustic is a ground for the corresponding curve in the class $C_{2*}(n,p)$ and base-line. This way it is possible to generate catacaustics which are also the envelopes of the circles centered on a Ribaucour curve and tangent to the base line.
Example of the caustic of the cycloid :

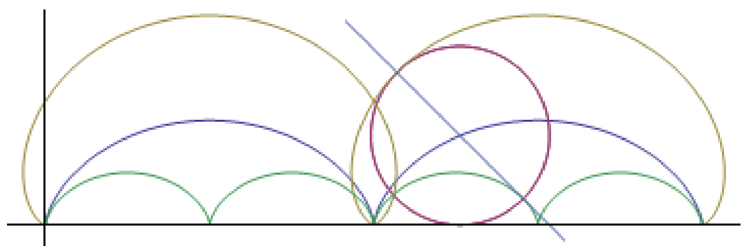

Figure 22: Y-axis parallele rays caustic of the Cycloid

7 A special class of Cesaro curves

Cesaro in (4) and Goormaghtigh in (8) study the curves such that the osculating circle dilated from the current point in a constant proportion k is tangent to a fixed line in the plane. These curves are just the one mentioned at the preceeding section and when algebraic are curves of direction.
This class of curves corresponds to Cesaro curves when $\lambda = \mu \neq 1$.
The ratio k is the inverse of corresponding index of the Ribaucour curve.
$\rightarrow$ k=1 is the circle since the homothety of ratio 1 is the identity : it keeps tangency with any tangent.
$\rightarrow$ k=-1 is the Tschirnhausen's cubic TC see (Part IV), the homothety is a symmetry wrt the current tangent. The symmetric of the osculator circle is tangent to the directrix of the TC.
$\rightarrow$ k=2/3 is the Nephroid. The locus of the centers of osculator circles dilated in the ratio 2/3 is the fixed circle and dilated circles are tangent to the diameter, classic property.
$\rightarrow$ k=1/2 is an evolute of the cycloid. The osculator circles dilated in the ratio 1/2 is tangent to the base/directrix of the initial cycloid.

$\rightarrow$ k=2 is Poleni's curve (or "courbe des forcats"). The osculator circle dilated in ratio 2 is tangent to the directrix.

This article is the 12th part on a total of 12 papers on Gregory's transformation and related topics.

Part I : Gegory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part II : A generalisation of sinusoidal spiral and Ribaucour curves.
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R. - Polar tangential and Axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.

There are two papers I have published in french :
Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (No 15 - Fevrier 1983).
Une generalisation de la roue - Bulletin de l'APMEP (No 364 juin1988).

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F. Gomez Teixeira Traite des courbes speciales remarquables Chelsea New York 1971 (3 tomes)
Nouvelles annales de mathematiques (1842-1927) Archives Gallica
Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
(1) E. Cesaro - Sur deux classes remarquables de lignes planes (NAM avril 1888 - 3 serie -Tome 4).
(2) E. Cesaro - Remarques sur la theorie des roulettes (NAM mai 1888 - 3 serie - Tome VII).
(3) G. Humbert - Sur les courbes algebriques planes rectifiables (JMPA - 4 serie - tome IV 1888).
(4) E. Cesaro - Sur une classe de courbes planes remarquables (NAM 1902).
(5) E. Turriere- Generalisation des courbes de Ribaucour (NAM 1913).
(6) F. Balitrand- Note sur les roulettes et les glissettes planes, a base rectiligne (NAM 1915).

(7) R. Goormaghtigh - Sur les familles de cercles (NAM - 4 serie - Tome XVI - janvier 1916).
(8) R. Goormaghtigh - Sur une classe de courbes planes remarquables (NAM - 4serie 1919 tome 19).
(9) http://www.mathcurve.com/ (Many fascinating informations and animations).
(A) - Nouvelles annales de mathematiques (1842-1927) Archives Gallica.
(B) - Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica.