CLOSED WHEELS AND PERIODIC GROUNDS Part - V

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Abstract

The property of Cardan-Al Tusi is the generic example of a couple wheel-ground as defined by the Gregory's Transformation. We study some characteristics of the couple wheel-ground for which the wheel and the corresponding ground are closed or periodic curves. We explore some examples to illustrate the topic just a glance through the subject.

1 The theorem of Cardan-Al Tusi

A theorem often attributed to Cardan - but known earlier by Al Tusi - expresses that a point O, on the outline of a circle-wheel rolling on a circle-ground of double size, runs along the diameter of the great circle. It is a couple ground-wheel as defined by the Gregory's Transformation. The wheel is the small circle $\rho = R.\cos\theta$ and the ground is the circle $x = R.\sin\theta$ and $y = R.\cos\theta$. More generally each point attached to the small circle describes an ellipse in the fixed plane. This ellipse collapses to the diameter travelled twice when the pole is on the small circle. This is Cardan's property that can also be seen as the two cusps hypo-cycloid.

All the points on the outline of the circle describe a diameter since all lines passing through the center are symmetry axis. This is a particularity of the circle.

We define wheels and grounds (see part I) as couples of rigid objects in the plane : the wheel = [a curve + a point : the pole O in the moving plane] and the ground = [a curve + a base-line Δ in the fixed plane] linked by Gregory's transformation. This paper presents a few examples of grounds and wheels that are closed curves.

We will study also the periodic grounds since rotation for the wheel and translation for the ground are dual elements.

2 The couple wheel-ground when the wheel is closed

We recall the direct GT ($y = \rho \quad x = \int \rho d\theta$) that associates to a wheel in polar coordinates the corresponding ground in cartesian coordinates. GT^{-1} is the reverse transformation : given the ground, find the wheel ($\rho = y$ and $\theta = \int_{xo}^{x} \frac{dx}{y}$).
The duality principle for wheels and grounds associates the pole O and the base-line (the x'x-axis) as dual elements. When the wheel is fixed the base line of the ground, rolling on the wheel, passes through the pole and when the ground is fixed the pole of the wheel, rolling on the ground, moves on the base line.
If the wheel passes through the pole then the ground crosses the base-line for $\rho = y = 0$.

To avoid complexity we examine only two types of closed curves :
1- The closed or periodic in x ground crosses orthogonally the base-line and is similar to the Cardan wheels.
2- The ground is entirely situated on one side of the base-line Δ, say in the upper half plane and is periodic of period T_x. This last case includes the closed grounds for T_x=0.

3 The closed wheel (C) is given

Two distinct cases are considered :
1- The pole is not on the wheel : $O \notin (C)$.
2- The wheel passes through the pole $O \in (C)$ and this point is not singular.

3.1 Pole $\notin (C)$

The wheel is closed and O is not on the outline. We examine two types :

3.1.1 Pole O is inside the closed wheel

Then corresponding ground doesn't cross the base-line and is on one side say the upper half plane Then the values of ρ are limited inside two intervals [ρ_{min},ρ_{max}] and $\theta \in [-\infty, +\infty]$. The ground is not closed and is a repeated periodic function from $x = -\infty$ to $+\infty$ inside a infinite strip limited by [$y = \rho_{min}$,$y = \rho_{max}$]. A period corresponds to one complete turn of the wheel.
An example for this type is the round wheel with O at the center and the ground is a line parallel to the base line. This sort of reversibility (ground same as roulette) is particular to this case. The pedal of the circle/wheel ($\rho = R \quad \forall\theta$) is the same circle. So in this case by Steiner-Habich theorem (see part I) the roulette and the ground are the same parallel to the base-line.

3.1.2 The wheel is seen from the Pole O with a limited angle of rotation

Then the values of ρ and θ are limited inside two intervals [ρ_{min},ρ_{max}] and [θ_{min},θ_{max}] so because $\rho \geq 0$ the ground is totally on one side of the base-line and is included in a strip defined by the first interval. In general the ground is a periodic motive that self repeats to $\pm\infty$.

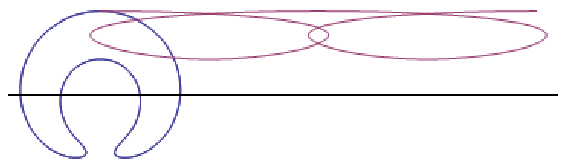

Figure 1: O not on the wheel : $\rho(t) = 0.5 + 0.2\cos(t)$ and $\theta(t) = 2.9\sin(t)$ and its periodic ground

The ground is closed if the following condition is verified :

$$\Delta x = Period_{ground} = \int_{t}^{t+1cycle} \rho.d\theta = 0$$

integrated on a complete cycle of the wheel. So the translation along x'x is 0 at the end of the cycle and the direct GT gives equations of a closed smooth curve. At the end of a turn the wheel is at the same position as at the beginning of the rolling (see fig. 2).

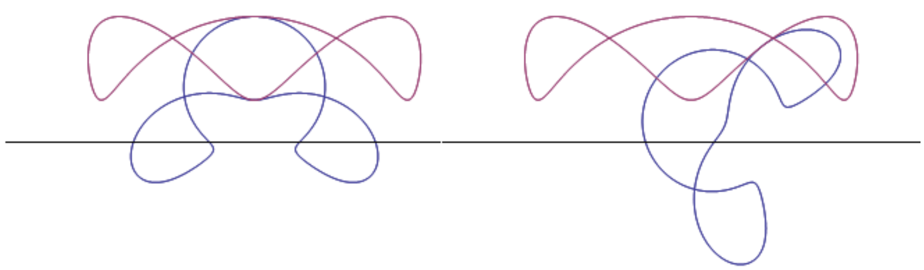

Figure 2: Pole outside the wheel - ground not crossing base-line (limited angle)
Wheel :$\rho = 2 + \cos 3t \qquad -2 \leq \theta = 2\sin t \leq +2$

3.2 Pole O $\in (C)$

The wheel is closed ($T_\theta = \pi$) - O on the outline.
Then the corresponding ground crosses the base line at two points and at these crossings the point O the wheel and the closed ground-curve are tangent.
Since the pole O is a smooth point : the ground crosses orthogonally the base-

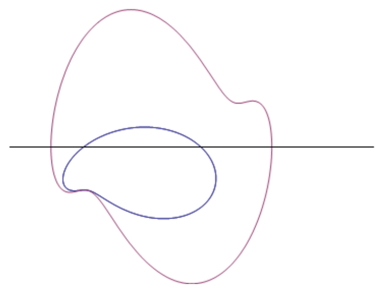

Figure 3: O on the wheel : $\rho = 3\cos^3\theta + \sin^3\theta$ and its ground

line at two points (Figure 3).
- The rolling is periodic and two turns of the wheel correspond to one course around the ground. After a turn rolling on the ground in the upper half plane the wheel crosses the base-line orthogonally and makes the symetrical movement in the lower half plane before returning in the initial position after one complete turn. The wheel turns twice inside the closed ground just as in the case of Cardan property. The pole O runs along a segment of the base-line between the two crossings.
These grounds have a center of symmetry on the base line at the mid point between the two crossings.
The length of the ground is twice the wheel's. The area inside the ground is four time the area inside the wheel.
If the wheel has an angular point at the pole O there is an angular gap in the rotation when the pole is on the base line at the crossing. (Example of the half-circle wheel with pole at the corner fig 19/3).

4 The periodic ground (Cr) is given

For a ground and its base-line x'x, the GT^{-1} gives the wheel in parametric equations from the ground equations : $\rho = y$ and $\theta = \int_{xo}^{x} \frac{dx}{y}$. When the ground crosses the base line y=0 with an angle $\neq \pi/2$ then there is a singularity for the wheel (angle goes to ∞ : it is similar to the logarithmic spiral and its asymptotic pole) and the wheel is not closed
So we limit the study to two cases :
1- The ground doesn't cross and is on one side of the base line.

2- The ground crosses orthogonally the base line at two not singular points.

4.1 The periodic ground is on one side of the base line

This types of grounds (see part 2) and the corresponding wheels can be explored by choosing parallele base-lines and varying the distance h to the ground. By a continuity argument it happens, as this distance varies, that the angle is increasing or decreasing and gives necessarily all integral number of turns and by GT^{-1} a closed wheel in polar parametric equations. We define a variable h = y offset of the base line ($y \to y \pm h$) to use the continuity argument.

4.2 Closed wheel for periodic ground along xx' axis. Period $= T_x$

The periodic ground includes the case of the closed ground if $T_x = 0$. Generally there is an angle offset after each turn rolling on a period of the ground and the wheel is not closed.
The condition on h for closure is : angular offset $= \frac{2k.\pi}{n}$

$$\int_{x_o}^{xo+t_p} \frac{dx}{y+h} = \frac{2k.\pi}{n}$$

In this solution the length of n periods of the ground is the same as the wheel's length. After an integer number n of turns rolling on the ground the wheel must be exactly in the same relative position w.r.t. the periodic ground. This equation permits to determine the values of $h(k,n)$ that give closed curves and by GT^{-1} supplies an infinite number of solutions if the integral of this angle is a rational number (k/n) $\in \mathbb{Q}$.

This question is related to a problem of L. Euler(1781) studied later by J.A. Serret (1852) : to find a curve with the same arc length as circle. GT^{-1} supplies a geometric solution using this last formula (see part 2).

4.3 Examples of closed wheel associated to periodic ground along xx' axis.

We examine periodic grounds that doesn't intersect the base-line and are completely in the upper half-plane.

To illustrate this case we examine the example of a cycloidal ground (base line parallel to the cusps line, period : $2.\pi$) :

4.3.1 The ground is a cycloid cusps upward.

$$x = t + \sin t \qquad y = h + 1 - \cos t = \rho$$

The distance h is between the tangent parallele to the cusps line and the parallele base line.

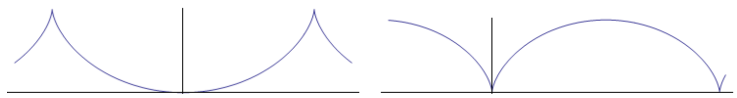

Figure 4: Cycloid cusps upward and cycloid cusps downward

$$\theta = -t + (2\sqrt{\frac{h+2}{h}} \arctan\left[\sqrt{\frac{h+2}{h}} \tan[t/2]\right])$$

The condition on d for the closure of the wheel is :

$$2(k/n).\pi = 2.\pi.\left[\sqrt{\frac{h+2}{h}} - 1\right] > 0 \rightarrow h = \frac{2n^2}{k^2 + 2nk}$$

Parameters n and k $\in \mathbb{N}$.

The fondamental (for n=k=1) is :

$$\rho = 5/3 - \cos t \qquad \theta = -t + 4.\arctan[2.\tan(t/2)]$$

As in the example studied in Part II for the circle-ground we have two sequences of closed curve. The first one the tangent.
The first sequence for the base line approaching to the tangent to the cycloid at the lowest point when n=1 and $k \in \mathbb{N}$ the curves have one cusp and a finite number k of small loops inside the big one. The second curve for k=2 is :

$$\rho = 5/4 - \cos t \qquad \theta = -t + 6.\arctan[3.\tan(t/2)]$$

The second sequence, for the base-line going away from the tangent for k=1 and $n \in \mathbb{N}$ the curves have a finite number n of cusps. The curve for n=2 is :

$$\rho = 13/5 - \cos t \qquad \theta = -t + 3.\arctan[(3/2).\tan(t/2)]$$

These two cases are included in the following formula with n and k as parameters (cycloid cusps upward) : $y = h + 1 - \cos t$ and $x = t - \sin t$

$$\rho = \frac{(\frac{n+k}{n})^2 + 1}{(\frac{n+k}{n})^2 - 1} - \cos t \qquad \theta = -t + 2\left[\frac{n+k}{n}\right].\arctan\left[\left(\frac{n+k}{n}\right).\tan(t/2)\right]$$

The base lines are under the x-axis and the parameters $n, k \in \mathbb{N}$. And we must not forget all intermediate cases between these two special sequences when n and k take all possible values in N.

When the base line is the tangent (for h=0) the above formula is not valid, the curve is the limit of the first preceding sequence with small loops and is not closed.The number k tends to ∞. A direct calculation gives :

$$\rho = 2\cos^2 t \qquad \theta = 2(t - \tan t)$$

It presents an asymptotic point at the pole (see fig.8).

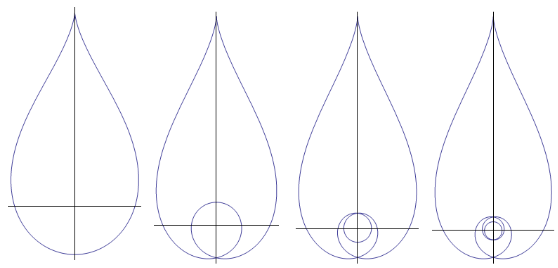

Figure 5: Wheels for base line under the arches for a cycloid with cusps upward (n=1, k=1, 2, 3, 4).

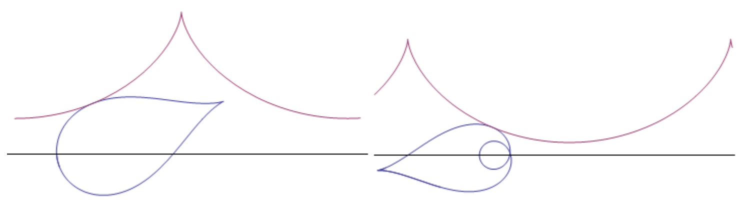

Figure 6: Fondamental wheel and first wheel with one inside loop for a cycloid cusps upward : n=1, k= 1, 2.

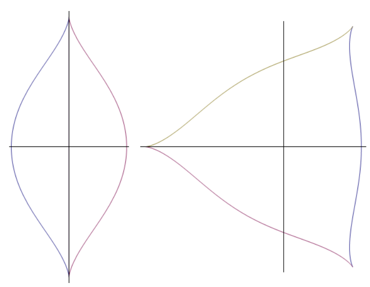

Figure 7: Wheels for base line under the arches for a cycloid with cusps upward (k=1, n=3, 4).

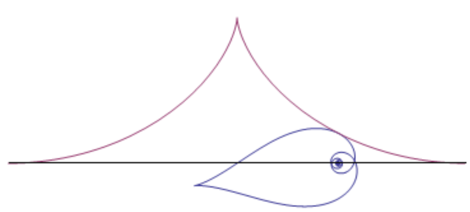

Figure 8: Limit wheel (base line =tangent) for the tangent when $k \to \infty$

4.3.2 The ground is a cycloid cusps downward (usual graph).

If the ground is the cycloid in the usual position cusp downward and the base-line parallel to the natural base :

$$x = t - \sin t \qquad y = d + 1 - \cos t = \rho$$

The distance d is between the cusps line and the parallele moving base line.

Inverse Gregory's transformation (GT^{-1}) gives :

$$\theta = t - 2\sqrt{\frac{d}{d+2}}\arctan\left[\sqrt{\frac{d+2}{d}}\tan[t/2]\right])$$

The condition on h for the closure of the wheel is :

$$2(k/n).\pi = 2.\pi.\left[1 - \sqrt{\frac{d}{d+2}}\right] > 0 \to h = \frac{2(k-n)^2}{2nk - k^2}$$

Values of n and k $\in \mathbb{Z}$.

The fundamental is the cardioid (n=k=1 and d=0) and can be found directly : $y = 1 - \cos t$ and $x = t - \sin t$ then $\rho = 1 - \cos t$ and $\theta = \int \frac{1-\cos t}{1-\cos t} dt = t$ so $\rho = 2.\sin^2(\theta/2)$. The base-line is the cusps line.
This time we have only one increasing sequence when k=1 is fixed and n $\in \mathbb{N}$
The second curve (for k=1, n=2) is :

$$\rho = 5/3 - \cos t \qquad \theta = t - \arctan[2.\tan(t/2)]$$

These cases are included in the following formula withh n and k as parameters (cycloid in usual position) : $y = d + 1 - \cos t$ and $x = t - \sin t$

$$\rho = \frac{1 + (\frac{n-k}{n})^2}{1 - (\frac{n-k}{n})^2} - \cos t \qquad \theta = t - 2\left[\frac{n-k}{n}\right].\arctan\left[\left(\frac{n}{n-k}\right).\tan(t/2)\right]$$

The base lines are under the x-axis and the parameters $n, k \in \mathbb{N}$. The formula is also valid if n=k (cardioid-wheel) since the arctangent is finite.

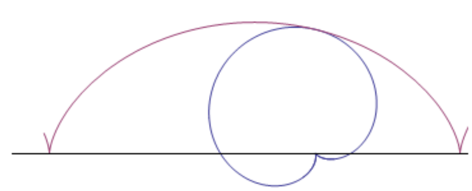

Figure 9: Fondamental cardioid-wheel for a cycloid cusps downward (n=1, k=1)

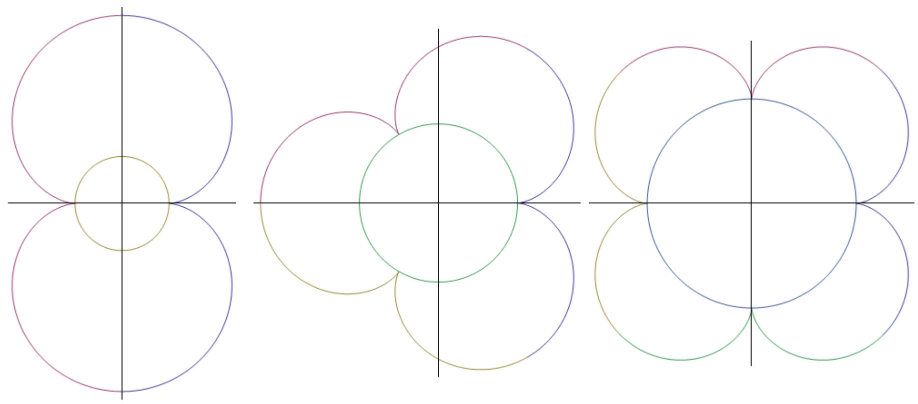

Figure 10: Wheels for base line under the cusps for a cycloid with cusps downward (k=1, n=2, 3, 4) - with different scales.

4.4 Ground defined by a curve in upper plane terminated at two ends on the base line with vertical tangents

We examine the only case when the ground is defined by a half curve in upper plane and which ends on the base line at two points with vertical tangent.
The ground intersects in two points x_o and x_1 and the base-line is a double normal.
If the following condition is true :

$$\theta = \int_{x_o}^{x_1} \frac{dx}{y} = \pm\pi$$

we can obtain a closed wheel by completing the upper part by an equal one obtained by central symmetry about the middle point of segment $(x_o + x_1)/2$ since after a turn of the wheel and a half turn inside the ground the two curves are in a symmetrical position in x_1 as at the beginning of the motion in x_o.
If the condition is not verified the ground, completed in the same way by symmetry, is also closed but there is an angular gap at the points xo and x1. This gap corresponds to an angular point for the wheel at the polar point O and rotations around the points x_1 or x_0. An example is showed on fig.19 ($n^o 3, 5$).
We are far from having exhausted the subject : there are many other possible cases.

5 Grounds and wheels made of arcs of lines, circles, catenaries and logarithmic spirals.

Some couples of associated specific simple curves permit to create wheels and ground we examine some of them : The wheel can be composed of miscellaneous

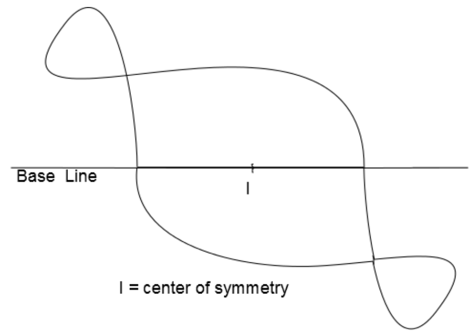

Figure 11: A central symmetric ground (without the wheel).

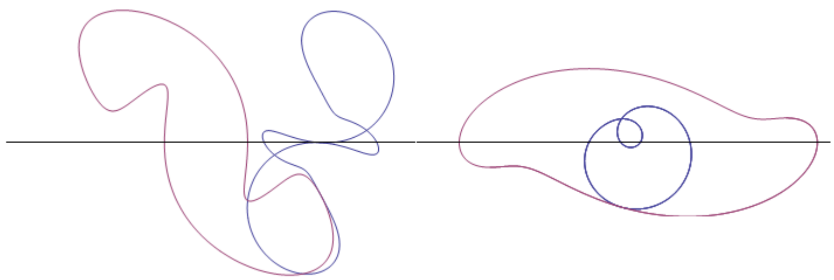

Figure 12: Exemples of central symmetric couples wheel-ground.

arcs of curves and a large diversity of couples wheel/ground can be constructed. We simply list couples of associated curves for wheels and grounds.

5.1 Polar stairs and cartesian stairs

GT have a connection with theory of integrals which is in the definition of Gregory's transformation and impose to the curves the same requirements of regularity as for integrable functions. In theory of integral simple discontinuities may be accepted like the stairs, the cusps, double points, angular points and this allows to construct non standard wheels.
A simple way to prospect thes possibilities is using stairs as indicated above with approximations of curves by stair-wheels. With arcs of circles centered at O and segment of lines passing through O. They correspond in the ground to segments of a line parallel to the base-line and segments of line orthogonal to the base.
Like for integrals we can approximate all wheels by polar stairs which are associated to ground stairs. To supress some exponential singularity on the base line a trick consists in using a stair so to cross (at a short distance) orthogonally the base-line (or critical line for the angle). For the wheel there is just a line passing through the pole O.

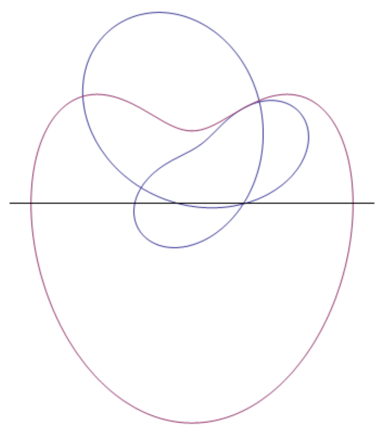

Figure 13: Wheel $\rho = 2\cos\theta - \cos 2.\theta$

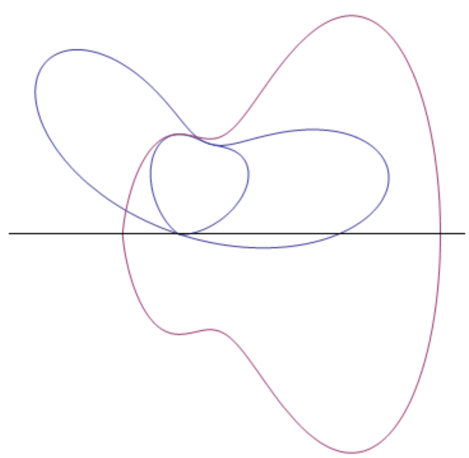

Figure 14: Wheel :
$\rho = \sin t - (1/2)\sin[2t] + (1/3)\sin[3t]$
$-(1/4)\sin[4t] + (1/5)\sin[5t] - (1/6)\sin[6t] + (1/7)\sin[7t] - (1/8)\sin[8t] \qquad \theta = t$

5.1.1 Arc of a line (W) - arc of a catenary (R)

This case is used to create polygonal wheels for grounds made up of arcs of catenary. For the ground there are two particular catenaries which are defined by the angle V:

- vertical lines orthogonal to the base-line ($V = 0$) or
- lines parallele to the base-line ($V = \pi/2$).

These are used to create stair-grounds.

5.1.2 Arc of Logarithmic spiral (W) - segment of Line(R)

This case helps to create polygonal grounds for wheels built with arcs of logarithmic spirals.
For the wheel we have seen above that there are two particular logarithmic spirals which are defined by their angle V:

- circles centred in O ($V = \pi/2$) or
- lines passing through O ($V = 0$).

These are used to create stair- wheels.

5.1.3 Arc of circle (W) - arc of double circle(R)

It is an application of Cardan property to create poly-arc wheels and grounds. These cases which combine the simplest forms to construct wheels or grounds may be computed by the direct or reverse Gregory's transformation. The next figures show some possibilities.

6 Examples of wheels and grounds

We give some illustrations a few case (among a wide set of possibilities) :

6.1 A Fourrier serie wheel :

$$\rho_k(\theta) = \sum_{n=1}^{n=k} \frac{1}{2n-1} \sin{(2n-1)\theta}$$

The case k=1 is Cardan property. This wheel "converges" if $k \to \infty$ to the example of half circle wheel rolling inside a rectangle of length π.

6.2 A half circle wheel.

It is possible to create a rotating piston using half circle (pole at the center) rotating in a rectangle : it rotates on the wall and is in the right place at each end of this rectangle. If the pole is at a corner or on the circle the ground is made of arcs of catenary and of circle.

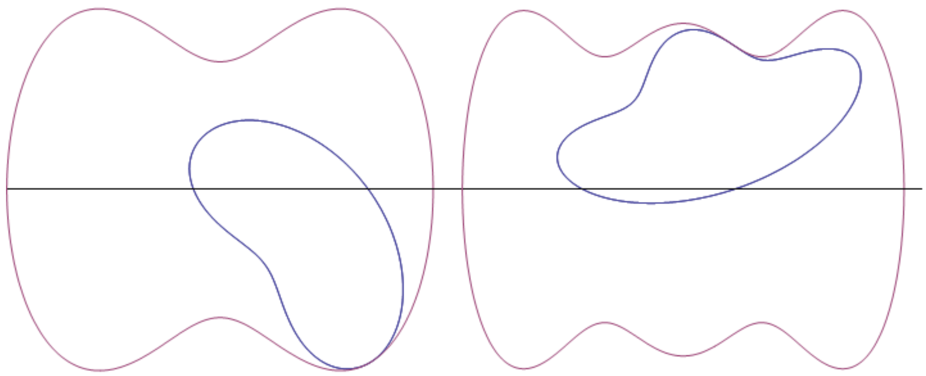

Figure 15: Fourrier wheel (k=2, 3)

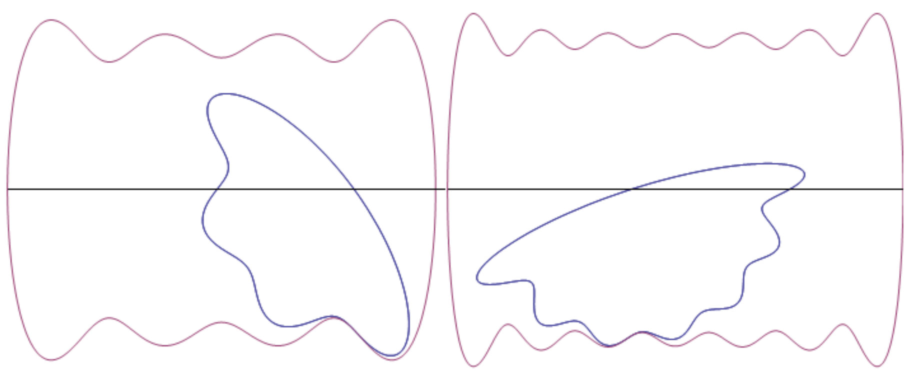

Figure 16: Fourrier wheel (k=4, 7)

6.3 A square wheel.

We take a square wheel with pole on the outline. Two cases are shown : pole at the corner and pole at the middle of a side.

6.4 A square ground.

The square has side a. We calculate the distance h between the base-line and the lower side of the square for the fondamental wheel (k=n=1) then :
$\alpha = a/(a+h)$ and $2.\pi.(1+\alpha).h = a$ so $h \approx 0,14a$ (see fig. 19/7).
For the general case we have :

$$\boxed{\pm\frac{k}{n}.2\pi = \frac{a}{h} - \frac{a}{a+h}}$$

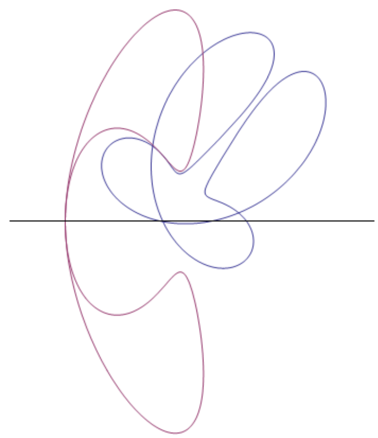

Figure 17: Doublepoint at the pole of the wheel and one ground with double orthogonal crossing of the base-line. Wheel with limited angle :
$\rho = 2\sin t + \cos 3t \qquad -1.55 \leq \theta = 1.55 \sin t \leq 1.55$

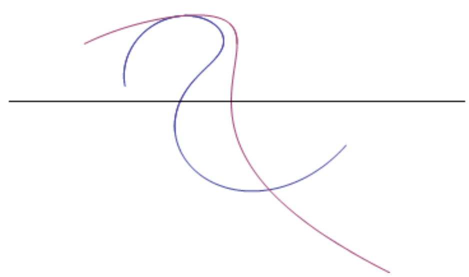

Figure 18: Doublepoint at the pole of the wheel and one ground with double orthogonal crossing of the base-line. Wheel with limited rotation : $\rho = 2\sin t + \cos 2t \qquad -1 \leq \theta = \cos 2t \leq 1$

6.5 A Fourier pediodic square crenel

The square has side a. We calculate the distance h between the base-line and the lower side of the square ground for the fondamental wheel (k=n=1) then : $\alpha = a/(a + h1)$ and $2.\pi = \frac{a}{a+h1} + \frac{a}{h1}$ so $h1 \approx 0,184.a$ (see fig. 19/8).
For the general case we have :

$$\boxed{\pm\frac{k}{n}.2\pi = \frac{a}{h1} + \frac{a}{a+h1}}$$

7 Conclusion.

We have used a line for the base-line but we can replace this line by a circle and another world is still to explore. There is a connection with the gregory's transformation because two wheels associated to the same ground - and two base-lines at distance d - are rolling curves for two poles at the same distance (see part I).
Fig 20 below shows an example a pole moving along a circle instead of a line. These examples could suggest it is possible to find mechanical applications or eventuallly to construct (why not ?) a real rotative engine (see fig.19-1).

This article is the 5th part on a total of 8 papers on Gregory's transformation and related topics.
Part I : Gegory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalisation of sinusoidal spirals and Ribaucour curves
Part IV : Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta-$curves.
Part VIII : Translations, rotations, orthogonal trjectories, differential equations, Gregory's transformation.
There are two papers I have published in french :
Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin1988).
References :
H. Brocard , T. Lemoine Courbes geometriques remarquables Blanchard Paris 1967 (3 tomes)
F. Gomez Teixeira Traite des courbes speciales remarquables Chelsea New York 1971 (3 tomes)
Nouvelles annales de mathematiques (1842-1927) Archives Gallica
Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
J Gregory Geometria pars universalis. Padova 1668.

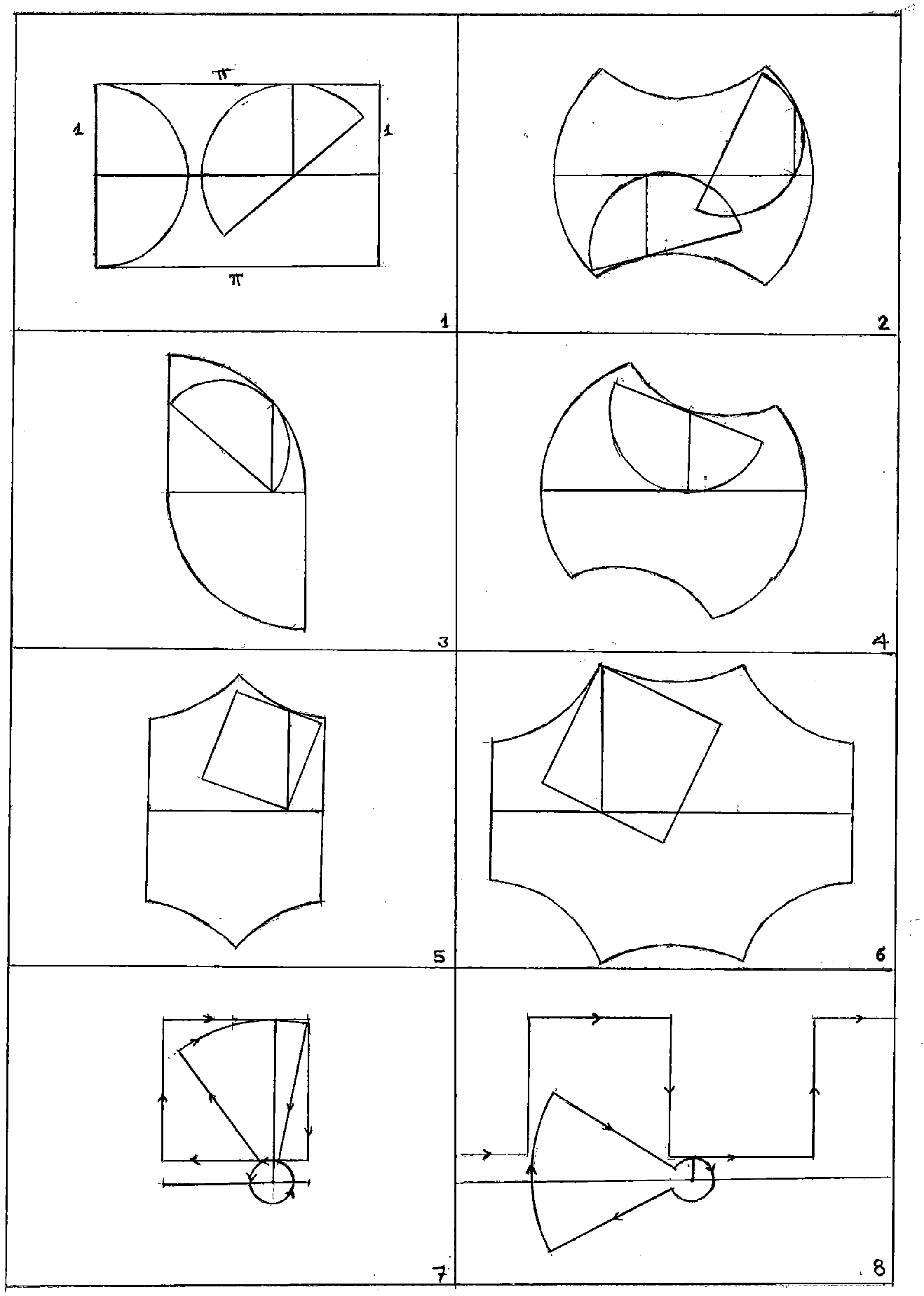

Figure 19: 1 to 4 : Half-circle wheel - pole at places on the outline - 1 is a rotative piston in a rectangle -, 5 / 6 : Square wheel - Pole at the corner (with angular point) and at the mid-side, 7 / 8 : Fundamental closed wheel for square ground and for a Fourier periodic crenel.

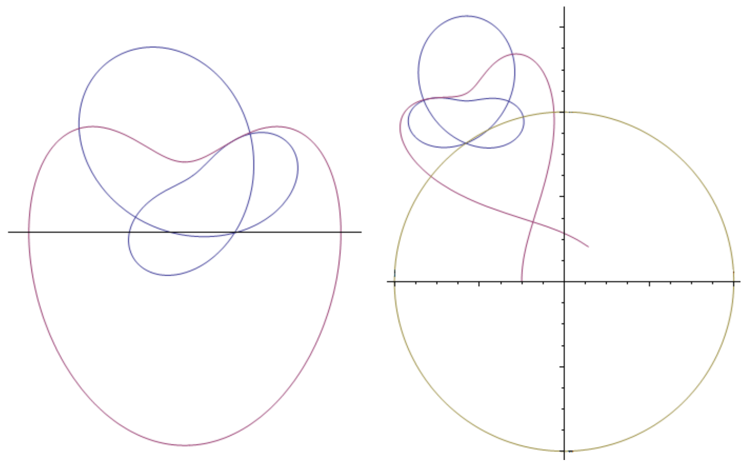

Figure 20: Pole moving on a line and pole moving on a circle for the same wheel : $\rho = 2\cos\theta - \cos 2.\theta$

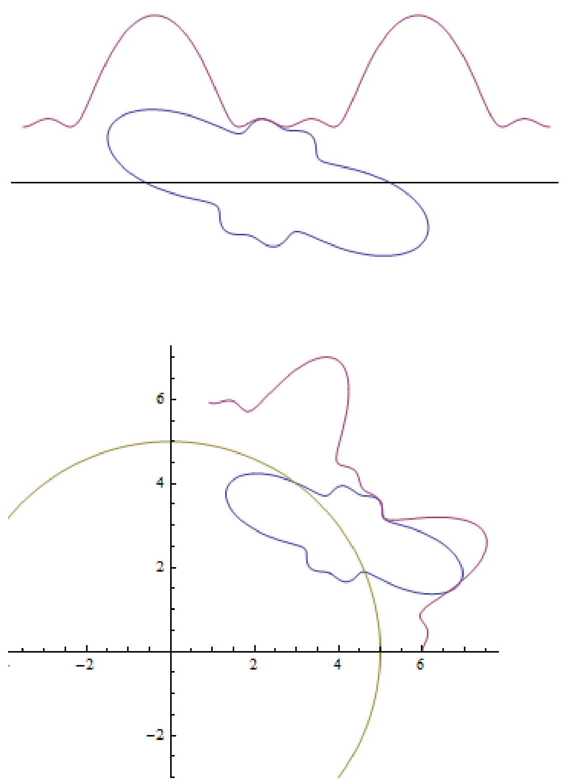

Figure 21: Pole moving on a line and pole moving on a circle for the same wheel : $\rho = 1 + (1 - \cos 2.\theta). \cos^2 2.\theta$