A GENERALISATION OF SINUSOIDAL SPIRALS AND RIBAUCOUR CURVES Part - III

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Abstract

We use two transformations Mc Laurin and pedal to define classes of plane curves that generalise the sinusoidal spirals. Since these spirals are wheels for the curves of Ribaucour, the properties of the couple wheel-ground associated by the Gregory's transformation in polar coordinates and in cartesian orthonormal coordinates give a simple generalisation. The common parameter of all curves is the angle V=(ray-tangent) or a multiple of this angle.

1 Plane curves with an arc length ds integrable by elementary functions

In the litterature about plane curves we can find classes of curves, like sinusoidal spirals $\rho = \sin^n(\theta/n)$, for which the expression the element of arc can be integrated by the mean of elementary functions.
If n is an integer the arc element $\sqrt{dx^2 + dy^2}$ of these spirals has no radical expression since the V is a multiple of θ. Most of curves have a $\sqrt{(.)}$ that is not possible to express by elementary functions as real or imaginary exponential (circular, hyperbolic, logarithmic and their reciproquals). This paper studies classes of curves that have elementary arc expressions.
Circular functions known since a long time have been used in geometry where one encounters often lines and circles.
The equation of the line is $y = k.x$ ($k = \frac{1}{\tan V}$) in orthonormal frame or $\rho = \frac{1}{\cos(\theta)}$ in polar coordinates then :

$$s = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1^2 + \frac{1}{\tan^2 V}} dx$$

$$= \int_a^b \sqrt{\frac{1}{\sin^2 V}} dx = \int_a^b \frac{dx}{\sin V} = (b-a)/\sin V$$

Another useful object is the circle with parametric equations (θ = angle at the center)

$$x = R\cos\theta \qquad y = R\sin\theta \qquad s = \int_0^\theta \sqrt{dx^2 + dy^2} = \sqrt{R^2}\theta = R\theta$$

The arc length of the line and of the circle are expressed by elementary functions and we search for classes of curves with the same property.

The sinusoidal spirals $\rho = \sin^p(\theta/p)$ or $\rho = \cos^p(\theta/p)$ present a simple relation between the angle V and polar angle : $V = \theta/p$ or $V = \theta/p + \pi/2$. Put $u = \theta/p$, and the polar parametric equations of the sinusoidal spirals are:

$$\rho(u) = sin^p u \quad \text{and} \quad \theta(u) = p.u \qquad V = u$$

V is the angle between the polar ray ρ and the tangent at the current point. We look for larger classes of plane curves with a similar linear relation between V and θ.

1.1 Two simple transformations : Mc Laurin of order n and pedal of order p

1.1.1 Mc Laurin's transformation :

Mc Laurin's transformation for curves in polar coordinates $(\rho\,,\theta)$ gives equations of transformed curve from the parametric polar equations of an initial curve. We suppose that $n \in \mathbb{Z}$, and set :

$$\rho_{McL} = \rho^n \quad \text{and} \quad \theta_{McL} = n.\theta$$

This is equivalent in transformation $Z = f(z) = z^n$ of the complex plane and it is a conformal transformation. So the angle V is preserved for the new curve. If $z = \rho\exp(\imath\theta)$ in the complex plane then $z^n = \rho^n\exp(\imath n\theta)$. We identify the complex plane $\mathbb{C}$ and the euclidean plane then the module $\rho_{McL} = \rho^n$ and $\theta_{McL} = n.\theta$.
For n=-n Mc Laurin's transformation is the inversion. In the complex plane this $z \to 1/z$ is an inversion associated with an axial symmetry. Beginning with the circle $\rho = \cos\theta$ or the line $\rho = 1/\cos\theta$ we get this way the sinusoidal spirals mentionned above.

1.1.2 Pedal transformation :

The pedal of a given a curve in polar coordinates is the curve dsecribed by the projection of pole O on the tangent at the current point of the first curve. The angle V is the same for two corresponding points of the curve and its pedal. The formulas to determine the parametric polar equations of the pedal of the curve (ρ_0, θ_0) given in polar coordinates, are:

$$\rho_1 = \rho_0 \sin V \text{ and } \theta_1 = \theta_0 - (\pi/2 - V)$$

The equation of order p pedal (p^{th}) is :

$$\rho_p = \rho_0 \sin^p V \text{ and } \theta_p = \theta_0 - p.(\pi/2 - V)$$

These curves are often defined up to a rotation around the pole O the angle $\pi/2$ may be forgotten. The anti pedal ($pedal^{-1}$) is defined in the same way so successives pedals or anti-pedals are defined in $\mathbb{Z}$. Multiply by $\sin V$ and substract V to the angle θ for the pedal, divide ρ by $\sin V$ and add V to the angle θ for the anti-pedal.
The circle $\rho = 1$ is self transformed by the Mc Laurin and the pedal transformations with the pole at the center of the circle.

1.2 Products of Mc Laurin and pedal transformations w.r.t. the same pole

Since the the angle V is common to all the curves it is possible to include the two transformations in a single formula by taking the p^{th} pedal of a n^{th} sinusoidal spiral and using u as the parameter we get :

$$\rho = \sin^n u . \sin^p u \qquad \theta = -nu - pu$$
$$\rho = \sin^{n+p} u \qquad \theta = -(n+p)u$$

If we apply these two transformations to the circle $\rho = \sin(\theta)$ we obtain the class of the sinusoidal spirals. But the two transformations keep the resulting curve inside the same family. So there is nothing new and it is necessary to add a new hypothesis. That's what we do now.

2 Curves with angle $\theta = n \tan u + p.u$

We noted that some well known curves like the involute of the circle :

$$\rho = \frac{1}{\cos u} \qquad \theta = \tan u - u$$

or its inversion w.r.t. to the pole (= tractrix spiral) :

$$\rho = \cos u \qquad \theta = \tan u - u$$

which are not sinusoidal spirals but have also angle $V = \pm u$, or even curves as the Sturm/Norwich spiral :

$$\rho = y = 1 + t^2 = \frac{1}{\cos^2 u} \text{ and } \theta = \tan u - 2u \rightarrow V = \pi/2 - 2u$$

$$\rho = y = 1 - \tan^2 u \text{ and } \theta = \tan u \rightarrow V = 2u - \pi/2$$

or $\rho = 1 - \theta^2$ all four curves are not included in the family of sinusoidal spirals so belongs to a larger class of curves.

This observation suggests to take the problem in a different way. We suppose that the angle θ in parametric polar equations is in the form : $\theta = n.\tan u + p.u$ and we calculate function ρ so to impose the angle V to be equal to the parameter u. This can be done by an elementary integration.

$$\tan V = \frac{\rho.d\theta}{d\rho} = \tan u$$

$$d\theta = [(n+p) + n\tan^2 u]du$$

$$\tan(u) = \frac{\rho[(n+p) + n\tan^2 u]du}{d\rho}$$

$$\int \frac{d\rho}{\rho} = \int \frac{(p + n + n\tan^2 u)}{\tan u} du$$

$$\int \frac{d\rho}{\rho} = \int (p+n)\frac{\cos u}{\sin u} + n\frac{\sin u}{\cos u} du$$

$$\ln \rho/C = (p+n)\ln\sin u - n\ln(\cos u)$$

We get the general parmetric polar equation (C=1) :

$$\rho = \frac{\sin^{(p+n)}(u)}{\cos^n u} \qquad \theta = n\tan u + p.u \qquad \text{and } V = u$$

These formulas depend on two parameters (n=Mc Laurin indice, p=pedal indice) and include the parametric polar equations of many curves as the Achimedean spiral (p=0, n=1), hyperbolic spiral (p=0, n=-1), spiral tractrix (p=1, n=-1), involute of the circle (p=-1, n=1) and also the family of sinusoidal spirals if (p=p, n=0). There is more information in the new equation $\theta = n.\tan u + p.u$.

3 Generalisation :$V = ku$ or $V' = \pi/2 - ku$ $(k \in Z)$

In this section we modify the requisition for the angle V of the curves and we impose $V = ku$ or $V' = \pi/2 - k.u$. The last case is $\tan V' = \frac{1}{\tan(\pi/2 - ku)}$. We use parameter angle u as the variable and by the same computation as above obtain solutions in polar parametric coordinates by integration of the following equation :

$$\tan V = \frac{\rho.d\theta}{d\rho} = \tan ku$$

Or

$$\frac{1}{\tan(V)} = \frac{\rho.d\theta}{d\rho} = \tan(\pi/2 - ku)$$

The integration of the preceding equations (for integers k, n, p) needs only elementary transcendental functions.

The general formulas are :

$C_k(n,p)$ for $V = ku$ and $\theta(u) = n\tan u + p.u$

$$\rho = e^{\int \frac{[(n+p)+n.(\tan u)^2]}{[\tan(k.u)]} du}$$

$C_{-k}(n,p)$for $V = \pi/2 - ku$ and $\theta = n\tan u + p.u$

$$\rho = e^{\int \frac{[(n+p)+n.(\tan u)^2]}{[\tan(\pi/2-k.u)]} du}$$

The results for the first values of the parameter k from 1 to 4 are listed below.

$k = 1$:

$$C_1(n,p) : \qquad \rho(u) = \tan^n u. \sin^p u$$

$$C_{-1}(n,p) : \qquad \rho(u) = \frac{e^{\frac{n}{2}\tan^2 u}}{\cos^p u}$$

$k = 2$:

$$C_2(n,p) : \qquad \rho(u) = \frac{(\sin u)^{(n+p)/2}}{(\cos u)^{(n-p)/2}}.e^{-n/4\tan^2 u}$$

$$C_{-2}(n,p) : \qquad \rho(u) = \frac{(\cos u)^{2n}}{(\cos 2u)^{(n+p/2)}}$$

$k = 3:$

$$C_3(n,p): \qquad \rho(u) = (\sin 3u)^{(n+p)/3}.\left[\frac{3-\tan^2 u}{\cos u}\right]^n$$

$$C_{-3}(n,p): \qquad \rho(u) = \frac{(\cos u)^{(4n/3)}}{(\cos 3u)^{(4n+3p)/9}}.e^{(n/6).\tan^2 u}$$

$k = 4:$

$$C_4(n,p): \quad \rho(u) = \frac{(\sin 4u)^{(n+p)/4}.(\cos 2u)^{n/4}}{(\cos u)^{3n/2}}.e^{(-n/8).\tan^2(u)}$$

The expressions for ρ are uniquely in terms of elementary functions of u but become quickly complicated. They represent the general curves $C_k(n,p)$. and have an element of arc length expressed by $ds = d\rho/\cos V = \rho d\theta/\sin V$.

The sinusoidal spirals are a special case inside the $C_k(n,p)$ because n=0 and p=p so $\rho = \cos^p(\theta/p)$ and $V = \theta/p$.
Three cases in the classes $C_k(n,p)$ are of particular interest :
- two for $V = ku: k = 1, k = 3$ and
- one for $V = \pi/2 - ku:$ k=2.

Indeed for these values of k the expression $\rho(u)$ can give - under conditions on n and p - only trigonometric functions (without real exponential).
We will see that these particular subclasses lead by the Gregory's transformation to grounds-curves that are linked to known curves.

3.1 Curves with angle $\theta = n\tan(u) + p.u$and $V = u$

These curves are related to the sinusoidal spirals and the curves of Ribaucour (CoR). In the XIX^{th} century the study of properties of these classes of plane curves was a classic subject (see on archives Gallica N.A.M. or J.M.P.A.). The sinusoidal spirals are wheels for the CoR w.r.t. to the natural base. A geometric definition of the CoR is the curves such that the part of the radius of curvature at the current point cuts the base-line in proportion $1/n$.
So n=1 gives the circle centered on the base line, $n = -1$ the catenary, $n = 2$ the cycloid, $n = -2$ the parabola, etc.
The corresponding sinusoidal spirals are for :
$n = 1$ The circle (theorem of Cardan-Al Tusi) $\rho = \cos\theta$,
$n = -1$ the line $\rho = 1/\cos\theta$,

$n = 2$ the cardioid $\rho = \cos^2(\theta/2)$,
$n = -2$ the parbola $\rho = 1/\cos^2(\theta/2)$ etc.
The general equation of these subclasses of curves wich I call $C_1(n, p)$, since k is fixed, depends on two rationnal parameters : n and p.

$$\boxed{\rho = \frac{\sin^{(p+n)}(u)}{\cos^n(u)} \qquad \theta = n\tan(u) + p.u}$$

The particular serie :

$$\rho = \cos^n u \qquad \theta = n.[\tan u - u]$$

are wheels for the following curves as the ground (these are the evolutes of the Ribaucour curves) :
n= 1 Spiral tractrix wheel corresponds to the tractrix (ground) :

$$y = 1/\cosh\alpha \qquad x = \alpha - \tanh\alpha \qquad (\sinh\alpha = \tan u \quad u = Gd\alpha)$$

the base-line is the asymptote.
n= 2 wheel corresponds to the cycloid (cusps upward) as the ground:

$$y = \cos^2 u \qquad x = \frac{\sin(2.u) - 2.u}{2}$$

the base-line is the tangent at the y minimum.
n= 3 wheel corresponds to the astroid ground :

$$y = \cos^3 u \qquad x = \sin^3 u$$

the base-line is a common cusp tangent.
n= 4 wheel corresponds to the ground-curve :

$$y = \cos^4 u \qquad x = \frac{\sin(4.u) - 4.u}{8}$$

n= -1 wheel (involute of circle) corresponds to the ground-curve :

$$y = \cosh\alpha \qquad x = \frac{\sinh(2.\alpha) - 2.\alpha}{4}$$

n= -2wheel corresponds to the ground-curve :

$$y = 1 + \sinh^2\alpha = \cosh^2\alpha \qquad x = \frac{2}{3}\sinh^3\alpha$$

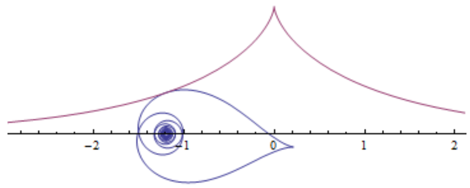

Figure 1: n=1 : Tractrix spiral - Tractrix

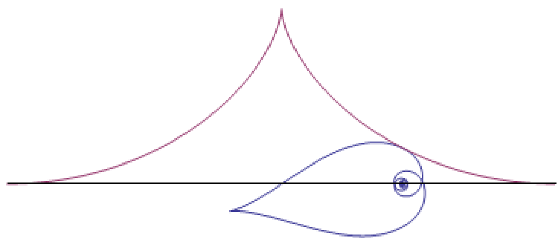

Figure 2: n=2 : Wheel for a Cycloid cusps at the top

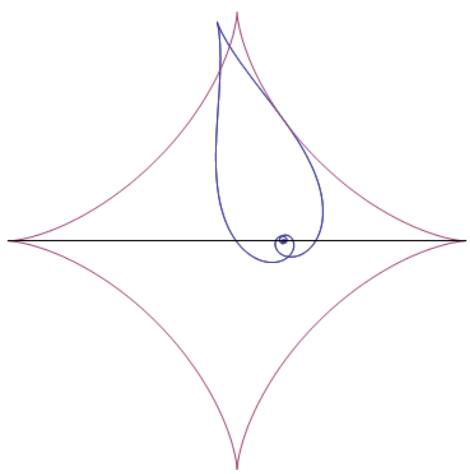

Figure 3: n=3 : Wheel for the Astroid

The particular serie (pedals of the preceeding serie) :

$$\rho = \sin u.[\cos u]^n \qquad \theta = [n \tan u - (n+1).u]$$

are wheels for the following curves as the ground :
n= 1 the wheel corresponds to the curve (ground) :

$$y = \sin u. \cos u \quad x = \ln[\cos u] - \sin^2 u$$

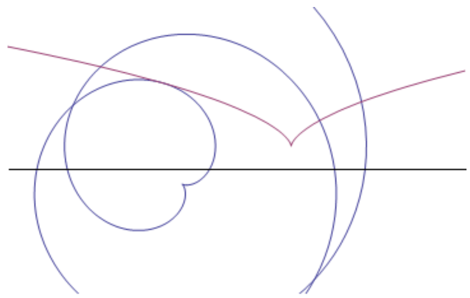

Figure 4: n=-1 : Wheel for the evolute of the Catenary = involute of the circle

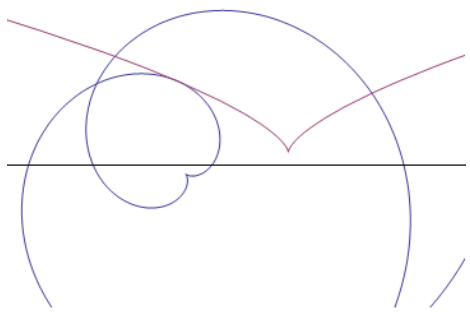

Figure 5: n=-2 : Wheel for the Semi-cubic parabola = evolute of the parbola

n= 2 the wheel corresponds to an involute of the astroid (ground):

$$y = \sin u.[\cos u]^2 \text{ and } x = \cos u.[1 + \sin^2 u]$$

n= 3 the wheel corresponds to the deltoid (ground) :

$$y = \sin u.[\cos u]^3 \text{ and } x = (1/2).\cos^2 u.[2\cos^2 -1]$$

n= 4 the wheel corresponds to following curve (ground):

$$y = \sin u.[\cos u]^4 \text{ and } x = (1/3)\cos^3 u(3\cos^2 u - 4)$$

3.2 Curves with angle $= n \tan u + p.u$ and $V = \pi/2 - 2u$

$$\tan V = \frac{\rho.d\theta}{d\rho} = \tan(\pi/2 - 2u) = 1/\tan(2u) = \frac{1 - \tan^2 u}{2\tan u}$$

$$d\theta = [(n + p) + n\tan^2(u)]du$$

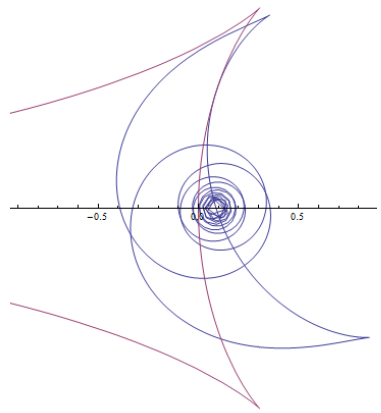

Figure 6: Wheel and ground : n=1

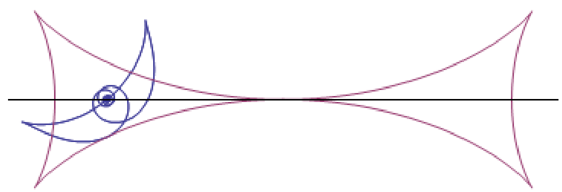

Figure 7: Wheel for special involute of Astroid : n=2

$$\frac{1-\tan^2 u}{2\tan u} = \frac{\rho[(n+p)+n\tan^2 u].du}{d\rho}$$

$$\int \frac{d\rho}{\rho} = \frac{2\tan u.[(n+p)+n\tan^2 u]}{1-\tan^2 u} du$$

$$\int \frac{d\rho}{\rho} = \int \left[n\frac{2\tan u.(1+\tan^2 u)}{1-\tan^2 u} + p\frac{2\tan u}{1-\tan^2 u}\right] du$$

$$\ln\rho = -n\ln\left[\frac{\cos(2u)}{\cos^2 u}\right] - \frac{p}{2}\ln\left[\cos 2u\right]$$

Then we have :

$$\rho = \frac{(\cos u)^{2n}}{(\cos 2u)^{(2n+p)/2}} \qquad\qquad \theta = n\tan u + p.u$$

We limit to integer power so replace p by an even number 2.p. This way p is exactly the pedal index since $V = \pi/2 - 2u$.

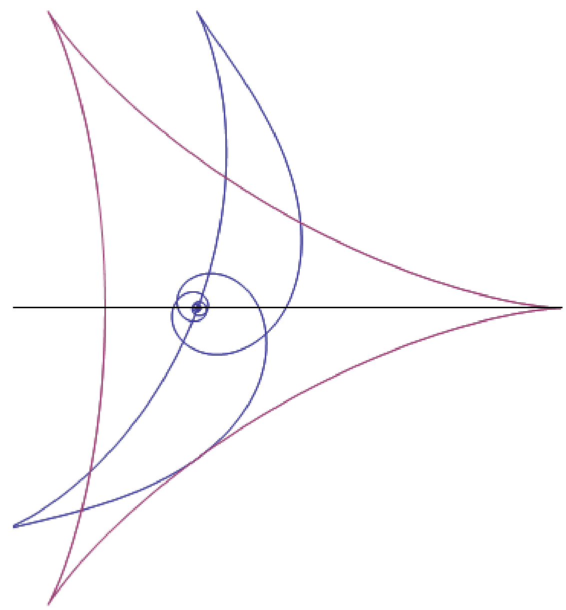

Figure 8: Wheel for the Deltoid-ground : n=3

$$\rho = \frac{(\cos u)^{2n}}{(\cos 2u)^{(n+p)}} \qquad \theta = n \tan u - 2p.u$$

Which can be rewritten to separate Mc Laurin and pedal parts , The equations of $C_{-2}(n,p)$ is :

$$\boxed{\rho = (\cos u)^{2n}.(\cos 2u)^{(p)} \quad \theta = n \tan u - 2(n+p).u}$$

3.2.1 Some particular wheels in this class $C_{-2}(n,p)$:

-1 The spiral of Sturm/Norwich $C_{-2}(-1,0)$:

$$\rho = \frac{1}{\cos^2 u} \qquad \theta = 2u - \tan u$$

this curve is such that $\rho = R_{curvature}$ and is an involute of the involute of the circle. $R_c = \rho$ as for the circle.
- 2 The wheel corresponding to a circle-ground w.r.t. to a tangent $C_{-2}(1,0)$:

$$\rho = \cos^2 u \qquad \theta = \tan u - 2u$$

this curve has a loop and an asymptotic point at the pole. It is the inverted of Sturm/Norwich spiral. Its length is s=u and intrinsic equation is :

$$R_c = 1/(3 - \tan^2 s)$$

and as same total lenth as the circle π and has many properties related to the circle (R=1/2).

- 3 The curve $C_{-2}(-1, 1)$ $\rho = \tan^2(u) - 1$ $\quad \theta = \tan u$ or $\rho = \theta^2 - 1$
- 4 The curve $C_{-2}(1, -1)$ $\rho = \frac{1}{1-\tan^2(u)}$ $\quad \theta = \tan(u)$ or $\rho = \frac{1}{1-\theta^2}$

this is the inverted w.r.t. the pole of the preceeding one and has the property $s = \rho.\theta = \frac{\theta}{1-\theta^2}$ and it has many related properties with the Line, the circle and the catenary.

-5 Wheel for the special syntractrix (Poleni's curve):

$$\rho = \cos u \qquad \theta = \frac{1}{2}[\tan u - 2u]$$

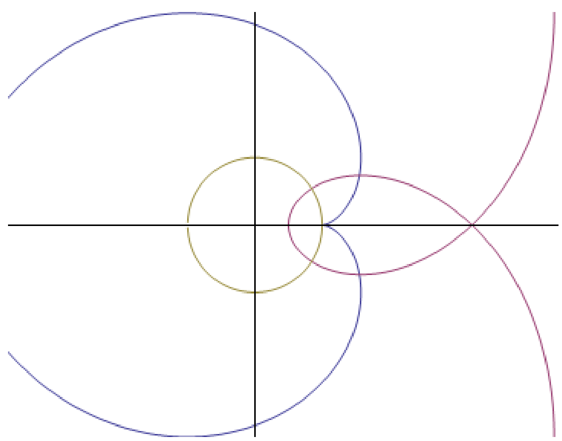

Figure 9: Circle, Evolute of the circle and Sturm/Norwich spiral : n=1

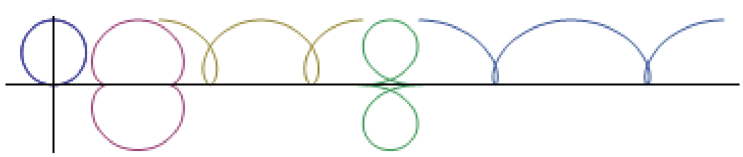

Figure 10: Circle, Nephroid, and three other "curves of direction" of the serie.

-6 Wheel for the L'Hopital quintic (or looping curve) :

$$\rho = 1/\cos^4 u \qquad \theta = 2[\tan u - 2u]$$

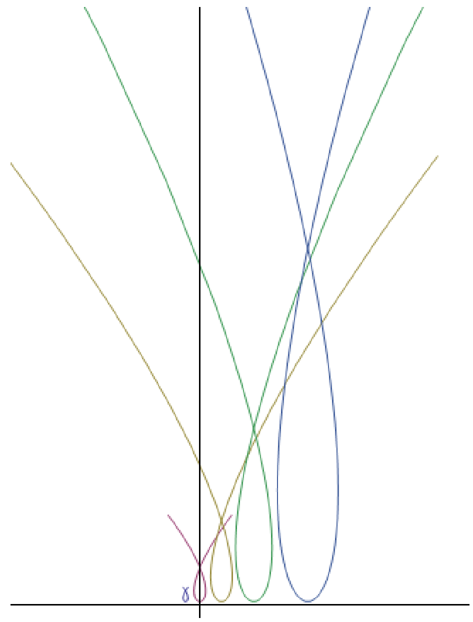

Figure 11: Tschirnhausen's Cubic, L'Hopital quintic and three other "curves of direction"of the serie.

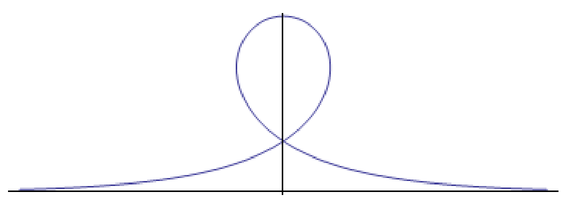

Figure 12: Poleni's curve

- 7 Wheel for the Nephroid (base-line is the cusp-tangent) :

$$\rho = \cos^3 u \qquad \theta = \frac{3}{2}[\tan u - 2u]$$

3.2.2 The corresponding grounds (base-lines x'x) for preceding curves are the following curves :

- 1 The Tschirnhausen's cubic :

$$x = t - t^3/3 \quad y = 1 + t^2$$

- 2 The Circle :

$$x = \sin\theta \qquad x = 1 + \cos\theta = 2\cos^2(\theta/2)$$

- 3 The Tschirnhausen's cubic :

$$x = t - t^3/3 \qquad y = t^2 - 1$$

- 4 The Double Catenary :

$$x = \int \rho d\theta = \int \frac{d\theta}{1-\theta^2} = \int \frac{1}{2}.\Big[\frac{1}{1-\theta} + \frac{1}{1+\theta}\Big]d\theta$$

We have two cases :
If $-1 < \theta < +1$ then (first ground) :

$$x = \frac{1}{2}.\ln\left(\frac{1+\theta}{1-\theta}\right) = \arg\tanh\theta = \alpha \qquad y = \cosh^2\alpha$$

If $\theta < -1$ or $\theta > +1$ then (second ground) :

$$x = \frac{1}{2}.\ln\left(\frac{\theta+1}{\theta-1}\right) = \arg\coth\theta = \alpha \qquad y = -\sinh^2\alpha$$

We have set :

$$\theta = \tanh\alpha \text{ then x=}\alpha \text{ and } \quad y = \cosh^2 x = \frac{1+\cosh(2.x)}{2}$$

and in the second case we put :

$$\theta = \coth\alpha \quad \text{ then x=}\alpha \text{ and } \quad y = -\sinh^2 x = \frac{1-\cosh(2.x)}{2}$$

We have two catenaries one with concavity upward and the other with concavity downward tangent to the base-line x'x. Note that there is an homothety of ratio 2 with catenary $y = \cosh x$.
- 5 Poleni's curve (particular syntractrix) w.r.t. to the asymptote as base-line :

$$x = 2.\tanh\alpha - \alpha \qquad y = 2/\cosh\alpha \qquad (\sinh\alpha = \tan u \quad u = Gd\alpha)$$

The radius of curvature and the $R_c = -\frac{1}{y}$ and has intrinsic equation: $R_c = -\frac{1}{2}.\cosh s$
- 6 The L'Hopital quintic (a special case of the looping curve) :

$$x = 2.[t - t^5/5] \qquad y = (1+t^2)^2 \qquad t = \tan u$$

- 7 The Nephroid :

$$x = \cos^3 u \qquad y = \frac{1}{2}(3 - 2\sin^2 u).\sin u$$

3.2.3 A subclass of curves related to "curves of direction" :

Les following curves:

$$C_{-2}(n,0) \quad : \rho = \cos^{2n} u \qquad \theta = n[\tan u - 2u]$$

presents interesting properties : they are wheels for "curves of direction" (Salmon-Laguerre). These are algebraic and have a rationnal arc so s is expressed by a rational formula in (x,y). Classic exemples are the Tschirnhausen's Cubic, the nephroid, the l'Hopital quintic. Some sinusoidal spirals :

$$\rho = \cos^{p/q}\left[\left(\frac{q}{p}\right).\theta\right] \text{ for odd p, q} \in \mathbb{N} \quad p \cap q = 1$$

are also "curve of direction".
The grounds corresponding to the class of wheels C_{-2}(n, p) are caustics by reflection for the rays of light coming from ∞ perpendicular to the base-line, i.e. the envelopes of the reflected rays of light on the initial curve or evolutes of the associated catacaustic. And these caustics are 'curves of direction' if they are algebraic. The subclass of grounds corresponding to curves C_{-2}(n, 0) - with 2.n$\in$ Z - are "curves of direction" (n=1 : Circle, 3/2 : Nephroid, -1 : Tschirnhausen's cubic, -2 : l'Hopital quintic).

3.3 Curves with angle $= n\tan(u) + p.u$ and $V = 3.u$

The expression of ρ for these curves is :

$$\rho(u) = (\sin u)^{-n}.(\cos u)^{-3n}.(\sin 3u)^{(4n+p)/3} = (\sin 3u)^{(n+p)/3}.\left[\frac{3-\tan^2 u}{\cos u}\right]^n$$

We limit to integer powers : $3|(4n+p)$ and we replace p by 3.p so p is the pedal index since $V = 3.u$. A subclass of the preceding solutions are the class $C_3(n,p)$. These can be written in the following way :

$$\boxed{\rho = \left[\frac{\cos u}{3-\tan^2 u}\right]^n.(\sin 3u)^p \quad \theta = n.\tan(u) - (n+3p).u}$$

Some are related to known curves :

$$C_3(1,0) \to \rho(u) = \frac{\cos(u)}{3-\tan^2 u} \text{ , } \theta = \tan u - u$$

This curve is the evolute of the wheel $C_{-2}(1,0)$ associated with the circle and a tangent as base-line : $\rho(u) = \cos^2 u \qquad \theta = \tan u - 2u$. If this curve $C_3(1,0)$ rolls on a line the pole O describes a cardioid, the line is its axis of symetry. By Steiner-Habich theorem the pedal of this last curve is a wheel for a cardioid as the ground and the base-line is the axis of symetry. So the parametric equations of the curve are p $\to p+1$:

$$C_3(1,1) \to \rho(u) = \cos^2 u.\sin 2u \text{ , } \theta = \tan u - 4u$$

is a wheel for the Cardioid (the base line is the axis of symetry of the cardioid). Its length is 4, $s = 2\sin u$. Its intrinsic equation is :

$$R_c^2.[7s^2-24]^2 = [4-s^2]^3$$

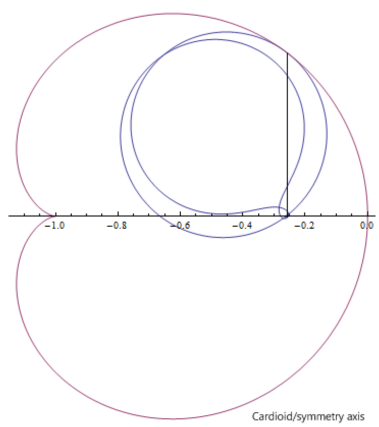

Figure 13: Wheel rolling on a Cardioid base-line is the symmetry axis

4 A Generalisation of the curves of Ribaucour :

In section 3.1 we have seen that sinusoidal spirals are wheels for curves of Ribaucour as the grounds so we get the equations of these

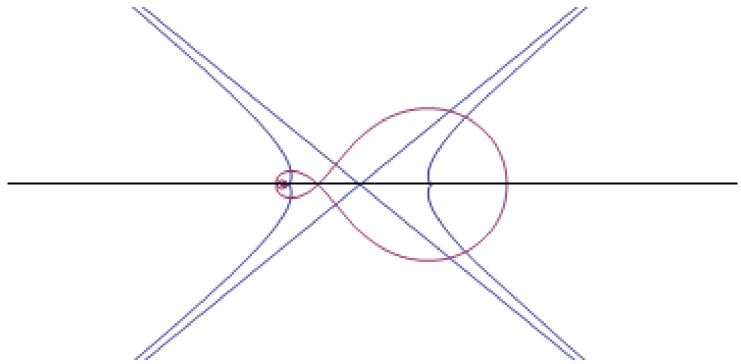

Figure 14: Wheel for the circle (base-line=tangent) and its evolute : the roulette of the pole is a cardioid

curves using direct Gregory's transformation.
This gives a possible definition of generalised curves of Ribaucour (GCoR): they are the grounds for $C_k(n,p)$ curves.
This definition includes the CoR since sinusoidal spiral are $C_1(0,p)$

5 Radius of curvature and arc length of the curves $C_k(n,p)$

The radius of curvature (R_c) of the curves $C_k(n,p)$ can be simply expressed if we use u as the parameter of the curve. We have: $ds = \frac{\rho d\theta}{\sin V}$, and $dV = kdu$ so the expression of the radius of curvature is :

$$R_c = \frac{ds}{d\theta + dV} = \frac{\rho}{\sin(V)} . \frac{d\theta}{[d\theta + k.du]}$$

And use the value of $\theta(u) = n.\tan(u) + p.u$ for particular cases.
The two last sections are speculative :

5.1 Cesaro's curves

The curves $C_k(n,p)$ seem to be special cases of a class of plane curves studied by E.Cesaro at the end of XIX^{th} century.These are curves which radius of curvature at M is proportionnal to the segment of normal between this point and the polar of this point w.r.t. a circle of radius R.

In an article of 1888 in Nouvelles Annales de Mathematiques he reviewed specific cases in relation to his equation and there might exist connections with the $C_k(n,p)$. The circle $\rho = R$ plays an important role among the classes of curves. Circles and conics are Cesaros's curves. Gomez-Teixeira found an equation involving ρ, R and V:

$$[\rho^2 - R^2]^{(n+1)} = a^{2n}.\rho^2. \sin^2 V$$

the polar equation :

$$d\theta = \frac{(\rho^2 - R^2)^{(n+1)/2}}{\rho.\sqrt{a^{2n}.\rho^2 - (\rho^2 - R^2)^{n+1}}} d\rho$$

and the equation for the radius of curvature:

$$R_{curv.} = \frac{a^n}{(n+1).\sqrt{(\rho^2 - R^2)^{n-1}}}$$

Two known subclasses of these curves correspond to R=0 (Sinusoidal spirals) and $R = \infty$ (Ribaucour curves). All intermediate cases, which are associated to $0 < R < \infty$, might be interesting.
The radius of curvature can only be infinite on the circle $\rho = R$ which can be cut only orthogonally by the Cesaro's curves so inflexions are on the circle $\rho = R$.

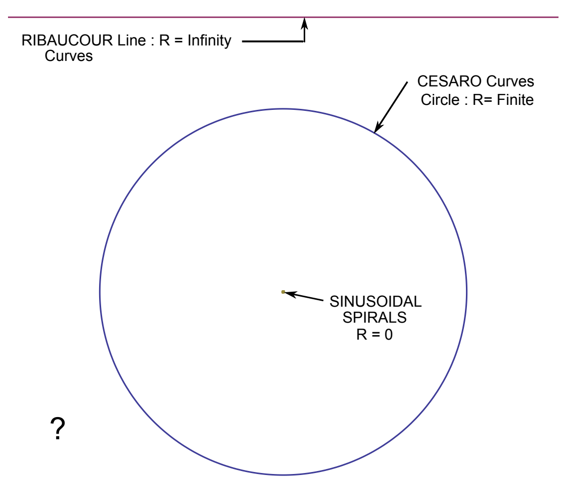

Figure 15: Sinusoidal spirals, Ribaucour curves and Cesaro's

6 The angle definition $\theta = n\tan(u) + p.u$

All the $C_k(n,p)$ curves are related to the circle and the line. The line with equation $\rho = \frac{1}{\cos(\theta)}$ and the circle with equation $\rho = 1$ or $\rho = \cos(\theta)$ are the starting curves for the construction of the classes $C_k(n,p)$.
The curvilinear abcisse on the circle depends on θ the one on the line depends on $\tan(\theta)$.
The equations of the evolute of the circle are $\rho = \frac{1}{\cos(u)}$, $\theta = \tan(u) - u$ illustrate the property : since the tangent rolls on the circle the polar angle is the difference between angle on the line tangent and angle at the center of the circle.

This article is the 3rd part on a total of 12 papers on Gregory's transformation and related topics.

Part I : Gegory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part II : A generalisation of sinusoidal spiral and Ribaucour curves.
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R. - Polar tangential and Axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.

There are two papers I have published in french :
Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (No 15 - Fevrier 1983)
Une generalisation de la roue - Bulletin de l'APMEP (No 364 juin1988).

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Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica

J. Gregory Geometriae pars universalis. Padova 1668.