Variable rolling circles, Orthogonal cycloidal trajectories, Envelopes of variable circles. - Part XIV - Draft-2

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Abstract

We recall some properties of cycloidal curves well known for being defined as roulettes of circles of fixed radius and present a generation using using variable circles as a generalisation of traditional roulettes and a different approach of these well known curves. We present two types of associated epi- and hypo-cycloids with orthogonality properties and give a new point of view at classical examples. We describe couples of associated cycloidals that can rotate inside a couple of cycloidal envelopes and stay constantly crossing at right angle.

1 Epicycles and Cycloidals :

Astronomy has used epicycles for a long time to describe, before Kepler and Newton, the motion of planet in the ptolemaic system. This theory of epicyles was not really explanatory and imposed to use the composition of circle motion (epicycles) to complete the description of the trajectory.
Rolling curves in the plane (or roulettes) used to study only curves assimilated to rigid objects. These roulettes, as the well known cycloid, were studied by many mathematicians of sixteenth century : Pascal, Huygens, Roemer, La Hire, Mc Laurin, and many others.
They present individually a great number of geometric properties and collectively other fascinating particularities generated by the rolling of circles or by envelopes of moving circles in the plane. We can find on the web many pages about cycloidal curves with supernatural properties isolated or collectively with special motions that seems nearly impossible - see examples on (9), (10), (11) web pages -. All are generated by only rotation and rolling of circles on other circles. The cycloidals generated, when algebraic - so generated as roulettes by two circles (fixed and rolling) with radii in a rational ratio - have wonderful geometrical specificities studied since a long

time for their surprising behavior.
In this paper we will often use the name "cycloidals" for all the curves either epi- or hypo-cycloids and are plane curves.

2 Complex of rotative cycloidal gears with tangential contacts.

A roulette is the trajectory of a point attached to a rolling circle (radius =r) on a nother fixed circle (radius = R). If the ratio = $R/r = m/n$ in $Q|$ m and n integers, then trajectories are closed algebraic cycloidals or trochoidal.
The very special properties of these cycloidals permits to generate the complex choregraphy of cycloidals moving tangentially on each other with cusps stay on their corresponding cycloidal and all seems to be a miracle. F. Morlay (1) in a paper (1894) gives a large overview of the phenomenon with many figures. M. Frechet (2) in a paper (1901) presents ellipses rolling inside deltoids . W. Wunderlich (4) generalizes to other trochoidals (1959). And P. Meyer (5) recalls these interesting references (1967), completing the picture including the extremal cases of the Cycloid (fixed circle is a line) and evolute of the circle (rolling circle is a line). Let us note that in these papers the curves are moving curves most often have a tangential contact with others and are sliding and not rolling without slipping.
We will see that there are also examples of moving cycloidals that maintain during the motion orthogonality at crossing points (instead of tangential contact).

3 Roulette of a variable circle rolling on a line.

We present now a sort of cycloid but the rolling circle has not a constant radius but is a function of a parmeter t. It is possible to define the motion of a point on the circle by an angle which is an integral of the elementary angle : $dx(t)/R(t)$.
A curve is defined in the plane in an orthonormal system (xx'Oyy') - by a variable circle rolling on the xx' axis with radius y=R(x) - or parametric: y = y(t) = R(t) and x=x(t), with usual hypothesis for the function (smooth or with isolated simple singularities). The value x is the integral of the radius of the variable circle length from x(to) and current point x(t) so we have the two relation :

$$\boxed{x(t) = s(t) = \int_{to}^{t} R(u)du} \quad (1)$$

$$\boxed{\theta(t) = \int_{xo}^{x} d\theta = \int_{xo}^{x} \frac{dx}{y} = \int_{to}^{t} \frac{dx(u)}{R(u)}} \quad (2)$$

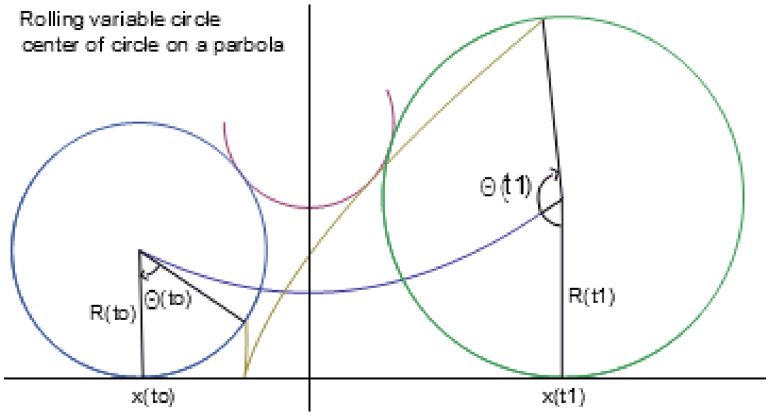

Figure 1: Trajectory of a point attached to a variable circle rolling on x-axis with center on a parabola.

so θ it is the cumulative angle of rotation of the variable circle and x(t) is the integral of the arc length of the variable circle between to and current point t. Note the analogy with equations defining Gregory's transformation (see Part I).

3.1 Roulette of a point attached to a variable circle.

We can define a generalized roulette on line x'x of a point sticked on the edge of a variable circle. The position of the point during the motion depends only of the cumulative angle indicated above. We say that the point is "angulary attached" and this means that the moving point is at distance R(t) from its center in the direction given by θ. The center of the circle is on the curve (x(t), y(t)=R(t)). The coordinates of the trajectories of attached points to this variable rolling circle are :

$$\boxed{X_M = x(t) - R(t)\sin\theta(t)} \qquad (3a)$$

$$\boxed{Y_M = R(t)[1 - \cos\theta(t)]} \qquad (3b)$$

We can consider these equations as a generalization of the cycloid, or as another way to look at Gregory's transformation for which the wheel has a variable radius and the trajectory of the center defines the curve.
This is a kind of polar system since s is indirectly defined by the angle θ and a length R(t) at the point of contact of the circle on the x-axis.
The formulas (1) for x(t) and (2) for $\theta(t)$ are complementary they just link the rotation to the translation in an integral form.
This case can be generalized to the rolling of variable circles or general roulettes of variable circles on any plane curve instead of the line like above.

The equations can be adapted to take in count the additional constraints, but are appreciably more complicated.

These equations are often used when the locus of the center is a given curve : [x(t), y(t)= R(t)], so only $\theta(t)$ needs to be computed.

3.2 The variable circle is centered on $x^2 + y^2 = 1$ and tangent to x-axis.

To illustrate this we shall examine a simple classical example that will show the way to a generalization. Suppose the center of variable circle stays on the circle $x^2 + y^2 = 1$ (or $x = \sin t$ and $y = \cos t$) and keeps tangential to x-axis. So $y = R(t) = \cos t$ and $x = \int_{to}^{t} R(u)du = \int_{to}^{t} \sin(u)du = \cos t - \cos to$. The parameter t is the polar angle at the center of the fixed circle. When t=to=0 the circle has radius 1 and is tangent to x-axis at O and when $t = \pi/2$ the circle with null radius is the points (-1,0) or (1,0). An elementary rotation of the variable circle is $d\theta = \frac{dx}{y}$. The rolling variable circle for intermediate position t=t, the angle is $\phi = \int_{0}^{t} \frac{dx(u)}{R(u)}$ so in this particular case (where x'(t)=y=R(t)): $\phi = \int_{to}^{t} \frac{\cos u}{\cos u} du = t - to$ which is just the length from starting point to the point of tangency of variable circle with the x'x-axis. At the begining the circle is at its maximum radius then R=1 and the angle of the position of the point attached to the variable rolling circle is $\phi = 0$ the position of this point when the variable circle rolls is given by the above equations for X_M, Y_M.
The equations in this case are the ones of a cadioid rotated with cusps on x-axis and passing through the points -1 and +1 on the same axis (see fig.1):

$$\boxed{X_M = \cos t \sin(t - to) + \sin t}$$

$$\boxed{Y_M = (1 - \cos(t - to)) \cos t}$$

The variable circle has for envelope two cycloidals : the nephroid and the 2-hypocycloidal that is also a part of x-axis or a flat ellipse (its total length is 4).
We will see it is possible to extend these facts to many configuration of cycloidal envelopes. And any smooth couples of curves envelopes tangentially generated by a variable circle which moves on a given curve in the plane as presented by J. Boyle in paper (7) for caustics by reflexion in the plane. These caustics can be generated by rolling variable circles. He gives among others the examples of the caustic of an ellipse, the astroid as the caustic of the deltoid, and of the Tschirnhausen's cubic as the caustic of the parabola

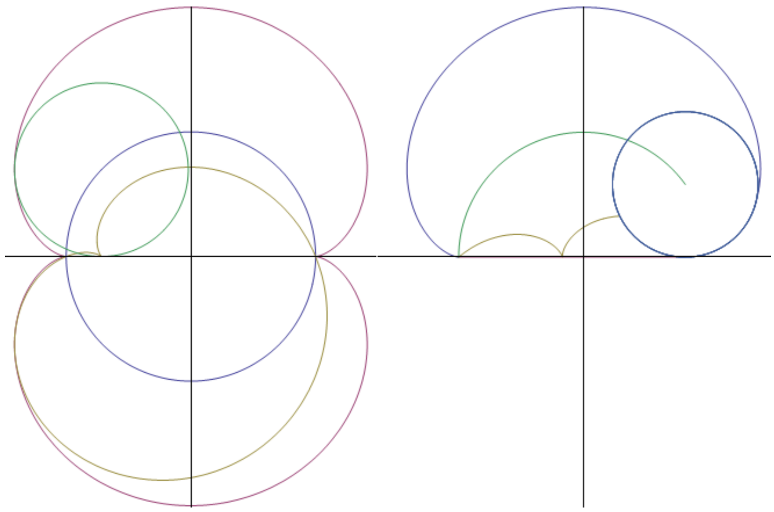

Figure 2: Trajectory of a point attached to a variable circle rolling on x-axis with center on circle $x^2 + y^2 = 1$. On the right the variable circle is represented at the current point (t=t).

when parallele light rays are coming from any direction.

We present now two types of associated cycloidals.

4 Couples of associated cycloidals envelopes with same cusps and same fixed base circle :

The first case of association of cycloidals is just the correspondance between an k-Epi- and an k-hypo-cycloids (k=common number of cups): The moving circle of same radius (r) can roll inside or outside the same fixed circle.The fixed circle (R) is called the base circle.
It is important to notice that, for each value of k (integer), there is an infinite number of epi-/Hypo-cycloids with k cusps.

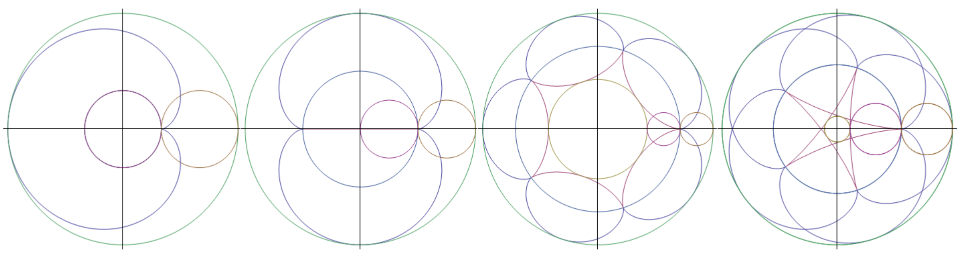

Figure 3: 4 couples of Epi-/Hypo-cycloidals with same cusps. Ponctually generated by circle (r) rolling on circle (R).

We suppose now that $m = R/r = p/q$ is the integer or rational ratio between the two circles radii.
These two cycloidals can also be generated tangentially by a variable circle. Its center moves on the fixed base circle at angle t from Ox and its radius is given by formula $R_v = 2.r.\sin(\frac{R.t}{r.2})$. Then it can be shown that the two points where the circle touches its envelope are on each of the two associated epi/hypo-cycloidals with same cusps. Points of contact are symetrical w.r.t. the tangent to the fixed circle at the center of the generating variable circle. This can be seen intuitively seen as the envelopes of a pulsating circle centered on a fixed base circle and moving with constant angular speed.

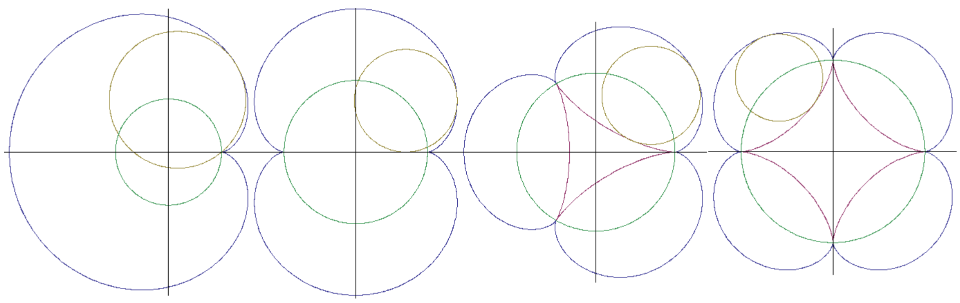

Figure 4: 4 couples of Epi-/Hypo-cycloidals with same cusps tangentially generated by circle $R_v = 2.r.\sin(\frac{R.t}{r.2})$ centeredon the base circle.

5 Couples of associated ortho-cycloids with same rolling circle inside a corona :

We present another way to associate two cycloidals.

A classical property of cycloids (trajectory of a point on a circle rolling on a line) is the following : The orthogonal trajectories of cycloids translated along their base line are equal cycloids generated by a circle rolling on the common tangent at the summits.

At an intersection point the two ERC (elementary rotation center) are on a same radius from this center and is a diameter of the same rolling circle. They are on each end of this diameter so the tangents are orthogonal at each intersection of corresponding cycloidals (fig.4).

In the next section we shall see this property can be generalized to cycloidals if we choose the appropiate couple of these curves.
We will below be interested by the trajectories of all points attached to the rolling variable circle.

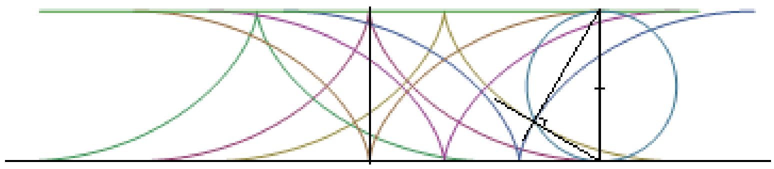

Figure 5: Couple :↔ Orthogonal cycloids for tanslation instead of rotation

5.1 Couples of linked Cycloidal curves as orthogonal profiles for rotation around O

This time we choose two concentric circles (m) and (n) of radii : m and n. We suppose m and n ($m > n$) are integers (or even rational numbers) and the third circle of radius r=(m-n)/2 (the locus of centers is the mid-circle (m+n)/2). The circle of radius r=(m-n)/2 can roll on each of the first circles, and any point on the edge will describe a k-epi- or k-hypo-cycloid entirely situated in the corona between the two circles ($R_1 = m, R_2 = n$). We consider the two cases where the moving circle rolls inside the great circle (m) or outside the small circle (n). In the first case the points describe an hypocycloid and an epicycloid in the last case. The two associated curves generated by the same rolling circle have an important property similar to the one above for cycloids - and translation parallele to the base - generalized to cycloidals. By the same argument used above for orthogonal cycloids and translation, we can prove that the two Epi-/Hypo-cycloids are orthogonal trajectories for the rotation around O.

5.2 Orthogonal epi- Hypo-cycloids in a corona:

In an orthonormed coordinate system x'xOy'y we choose three fixed circles (R) and (R±r) centered at O - and (r) - the rolling inside the corona between the two circles (R±r). We consider two motions : (r) rolling on and inside (R+r) generating an Hypocycloid and (r) rolling on and outside (R-r) generating an Epicycloid. The fixed circle (R) is the base circle. All points on circle (r) when rolling without slipping describe the same epi- or hypo-cycloids just rotated by any angle around O.
We choose a rational number $m = p/q$ with $p, q \in Z/p \cap q = 1$ the parameter that is the caracteristic of the our special cycloidals. We define the fixed base circle of radius m centered at O in the system of axes x'xOy'y. We define two other fixed circles center in O : (C1) of radius : $m - 1$ and (C2) of radius $m + 1$ and two couples of rolling circles : one of radii 1 with center on the fixed circle. This last circle will roll - externally - on (C1) generating

an ⊥-epicycloid and -internally- on (C2) generating an ⊥-hypo-cycloid. The Hypocycloid has (p+q) cusps and the Epicycloid $\mid p-q \mid$ cusps. The same result can be obtained using the alternative generation of Lahire. Since the inverse values m and 1/m lead to the same geometric results up to a dilation (1/m) from O, we will keep m.
The equations of these associated cycloidals are :

$$\boxed{X_1(t) = m\cos t + \cos[-mt]}$$

$$\boxed{Y_1(t) = m\sin t - \sin[-mt]}$$

$$\boxed{X_2(t) = m\cos t + \cos[-mt]}$$

$$\boxed{Y_2(t) = m\sin t + \sin[-mt]}$$

For m=1/1 : ⊥-cycloidals are a right segment and a circle turning inside a cardioid and passing through the cusp,
For m=2/1 or 1/2 : ⊥-cycloidals are the cardioid and the deltoid turning inside 2-epi/2-hypocycloids,
For m=3/1 or 1/3 : ⊥-cycloidals are the nephroid and the astroid turning inside a 3-epi/3-hypocycloids,
For m=p/q or q/p : ⊥-cycloidals are the (p-q)-epi and (p+q)-hypocycloids,
The two numbers for each generation correspond to the two Lahire generations of the cycloidals.

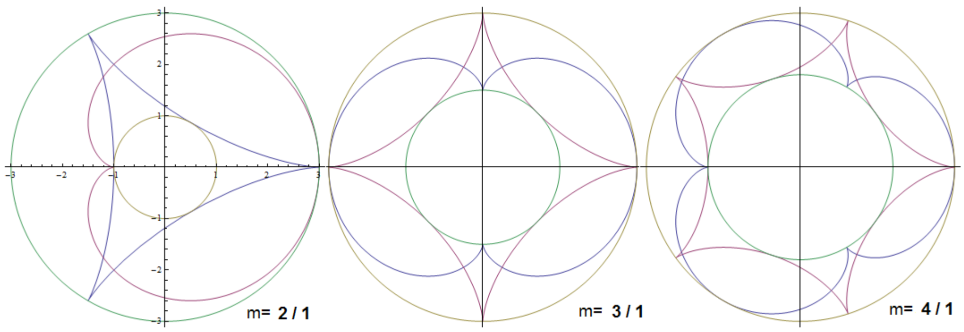

Figure 6: Couples : Cardioid/Deltoid (m=2/1), Nephroid/Astroid (m=3/1) and 3-Epi/5-Hypocycloid (m=4/1)

6 Couples of same cusps cycloids and rolling variable circle

We consider first the case of the classical cycloid generated by a point on a circle rolling on a line. Two cycloids with same cusps and situated on

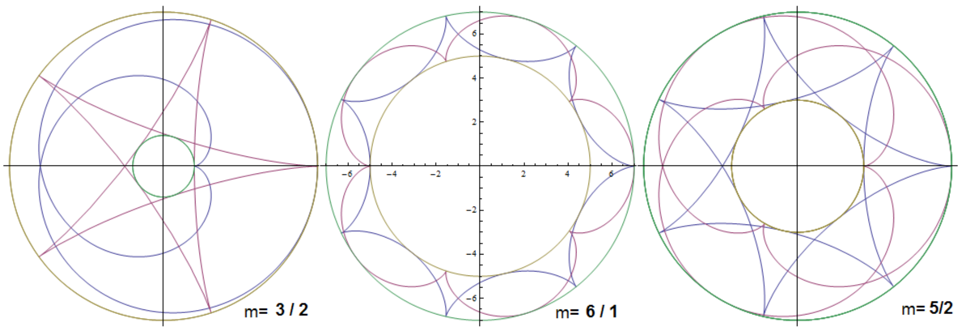

Figure 7: Couples m=3/2, 6/1 and 5/2

each side of this line. The rolling circle is the variable circle centered on the line and tangent to the two cycloids. If we impose that the variable circle rolls on, say, the lower cycloid a point angulary fixed to the circle will describe a cycloid translated with its cups on the lower cycloid and tangent to the upper one. This cycloid will pass through the common cusps of the two given cycloids. Equations of the two fixed envelope cycloidals (up and

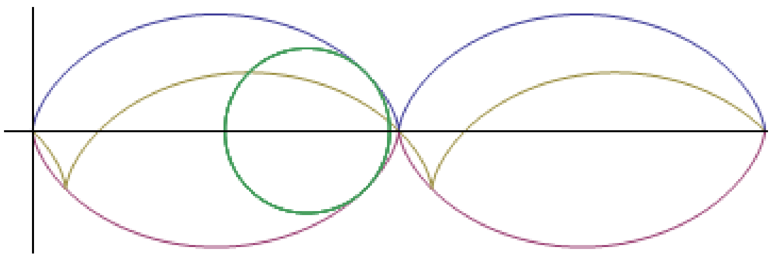

Figure 8: Trajectory of a point angulary fixed to the variable circle $\leftrightarrow$ translated of one of the inital cycloids.

down) with same cusps are :

$$x_1(t) = R.t - R\sin t; \qquad y_1(t) = R - R\cos t$$
$$x_2(t) = R.t - R\sin t; \qquad y_2(t) = -R + R\cos t = -y_1(t)$$

Equations of the locus of the variable circle at position t_p are cycloidals (3) and (4):

$$x_3(t) = R.t + 2.R\sin(t/2)\cos((t - t_p)/2)$$
$$y_3(t) = -2.R\sin(t/2)\sin((t - t_p)/2) \qquad (3)$$
$$x_4(t) = R.(-t) - 2.R\sin(-t/2)\cos((-t + t_p)/2) + 4\pi$$
$$y_4(t) = 2.R\sin(-t/2)\sin((-t + t_p)/2) \qquad (4)$$

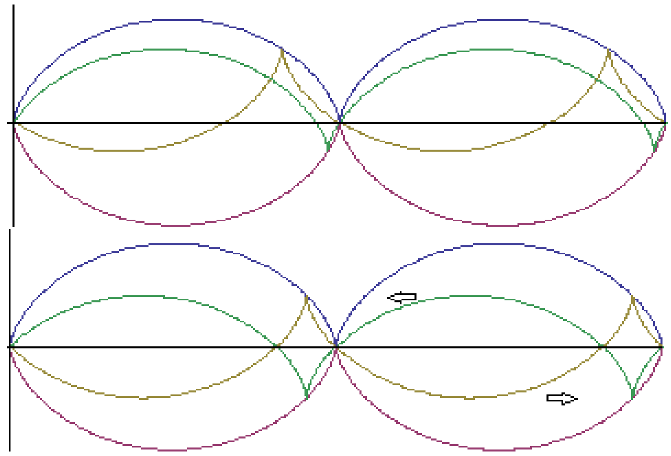

Figure 9: Couple :↔ Case of Orthogonal cycloids for tanslation. The two cycloids translate in opposite direction and meet orthogonally at cusps.

The cycloidals (3) and (4) above translate, during the motion, in opposite direction in such a way that they stay in the space between the same cusps cycloidals and are constantly ortogonal when they pass through these fixed cusps. And we will see that this property can be generalized to the associated orthogonal cycloidals.

We shall see this property can be extended to all cycloidals generated by a circle rolling on another fixed circle.
Note that at the cusps on one of the cycloidal the tangent is not normal to this curve , because the radius is varying, as can be seen on the figures.
The curves generated by points on the variable circles are moving curves inside the space between the two associated envelope cycloidals and pass through the common cusps where the circle has radius = 0.

7 An example of Rolling variable circles : Line - Nephroid - Cardioid -Deltoid

The nephroid is the envelope of a variable circle centered on a fixed circle and tangent to a diameter. This circle rolling on x'x (=2-hypocycloid or diameter) generates a cardioid rotating inside the nephroid (=2-epicycloid) and passing through the cusps of the nephroid. If the variable circles rolls on the nephroid this time we get a deltoid passing also through the cusps of the nephroid keeping its 3 cusps on it.

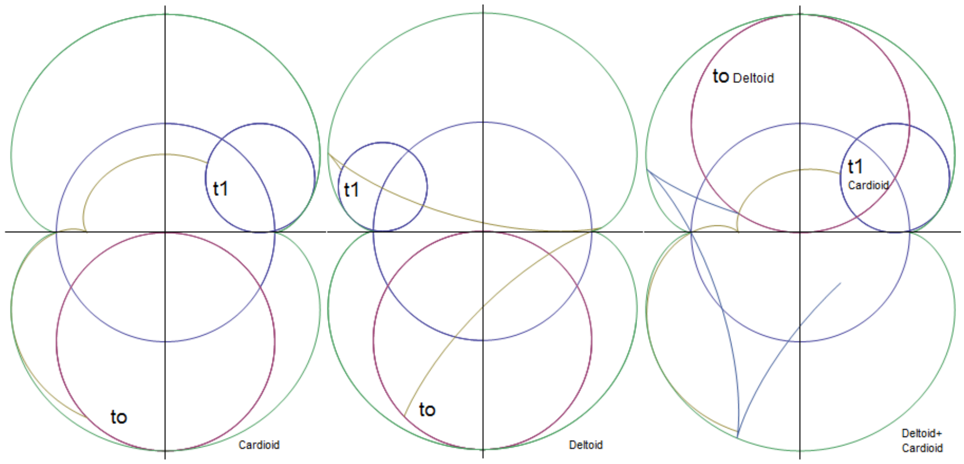

Figure 10: Rolling variable circle generating cardioid and deltoid inside a nephroid.

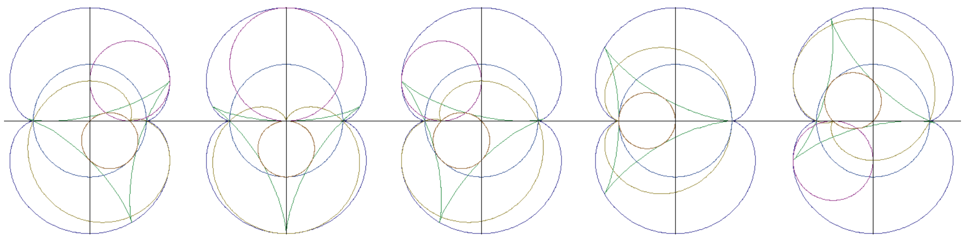

Figure 11: Orthogonal cardioid and deltoid turning inside a nephroid. During the rotation the two moving cycloidals are orthogonal at the fixed cusps of fixed envelopes cycloidals.

7.1 Curves generated by rolling circles or envelopes of circles:

In (7) J. Boyle proves interesting theorems about the generation of caustics in the plane and the double envelopes of familes of circles. His paper presents among others properties, the following :
"- Given a caustic Envelope (E) resulting from a given reflective curve $\alpha(s)$ and radiant S there is a curve $\beta(s)$ and a family of circles C_s that roll on β without slipping such that each circle has a point that will trace the caustic envelope as the circle roll".
Caustics by reflection for parallele light rays can be generated by variable rolling circles (the discriminant circle with radius $\frac{1}{4}R_c$ where R_c is the radius of curvature of the current point of the reflecting curve.

Nota : This idea of J. Boyle in (7) is a generalization of the "roulettes" to a variable circle in a simple way using an integral angle defined by the

ratio of two lengths $\frac{ds}{R}$.
R. Goormaghtigh (3) knew the interest of envelopes of families of circles. He gave some theorems about these families from an initial theorem of Cesaro (12) on the alignment of the centers of curvatures of thres curves : the one decribed by the center of the variable circle (R=R(s)) and its two envelopes. It would interesting to understand what is the geometric relation between these two envelopes. Abel Transon's concept of deviation with a logarithmic spiral is at the heart of the solution since the tangent point with second envelope of the variable circle rolling on a curve coincides with the pole of the osculating spiral - see (Part XVI)-.

Theorem (E. Cesaro - 1900) : If the variable circle touches its envelope at the two points K_1 and K_2 then the centers of curvature I_1 and I_2 at these two points are on a line that pass through the point where the envelope of K_1K_2 touches its envelope. And this point devides I_1I_2 in the ratio of the radii of curvature of the loci of K_1 and K_2.

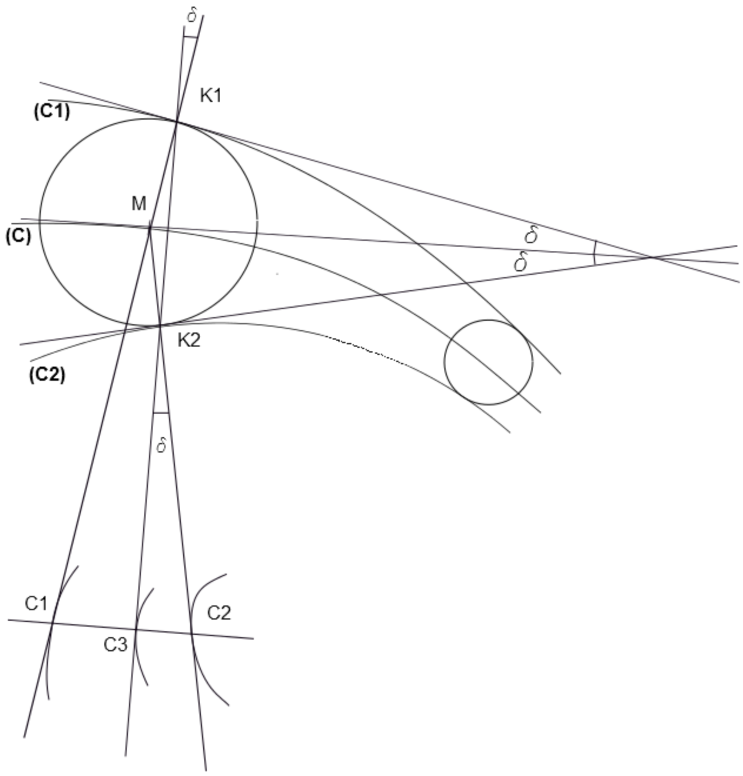

Figure 12: Envelopes of families of variable circles with center on a given curve C $\leftrightarrow$ Cesaro's theorem on the centers of curvature.

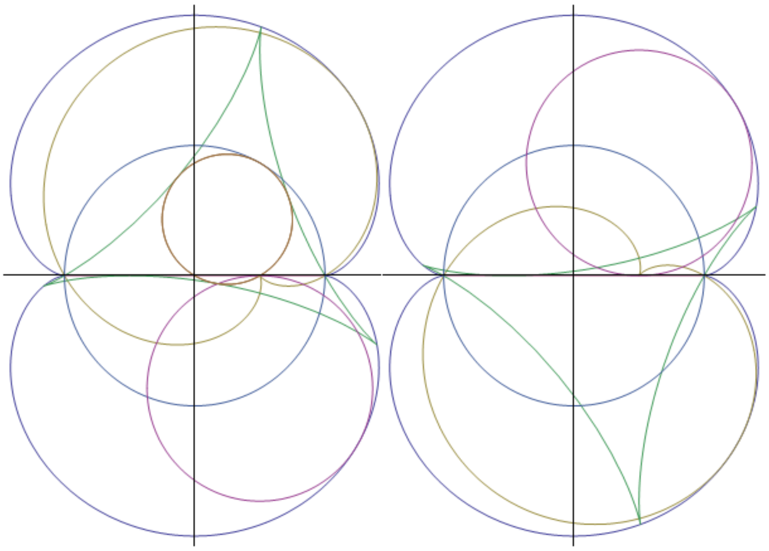

Figure 13: Trajectory of a point angulary fixed to the variable circle $\leftrightarrow$ rotating orthogonal Cardioid - Deltoid

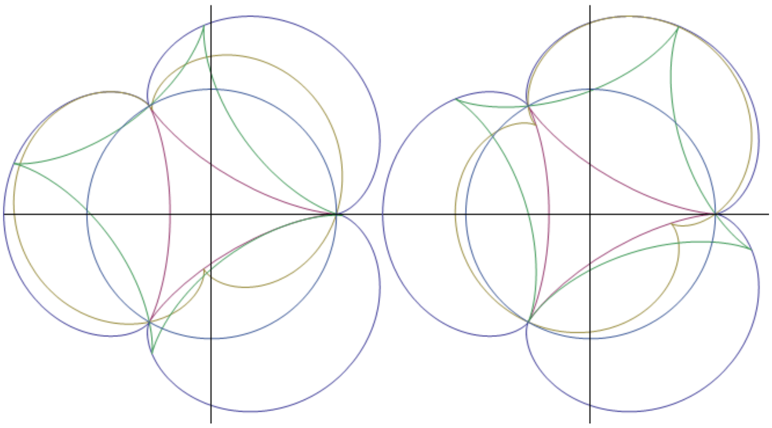

Figure 14: Trajectory of a point angulary fixed to the variable circle $\leftrightarrow$ rotating Astroid, and Nephroid.

And the property can be generalized to all the similar couples of associated orthogonal epi-/hypo-cycloids turning inside a couple of k-epi-/k-hypo-envelopes cycloidals.

8 Couples of same cusps epi/hypo-cycloids and rolling variable circle

We use the variable circles centered on the fixed circle of cycloidals and tangent to the k-epicycloid and the k-hypocycloids. When the circle rolls without slipping on one of these curves a point "angulary" fixed to the variable circle describes an epicycloid when rolling on the hypocycloid and an hypocycloid when rolling on the epicycloid. It is important to notice that the variable circles turn around in opposite angular direction.

8.1 Couples cardioid/point envelopes and circle/line as ortho-cycloidals and rolling variable circle

In this example one of the envelopes is reduced to a point (the cusp of the cardioid) and the other is a cardioid. The two ortho-cycloid are a circle passing through the cusp and tangent to the cardioid and a right segment passing at the cusp, and at the center of the circle and keeping ends on the cardioid.

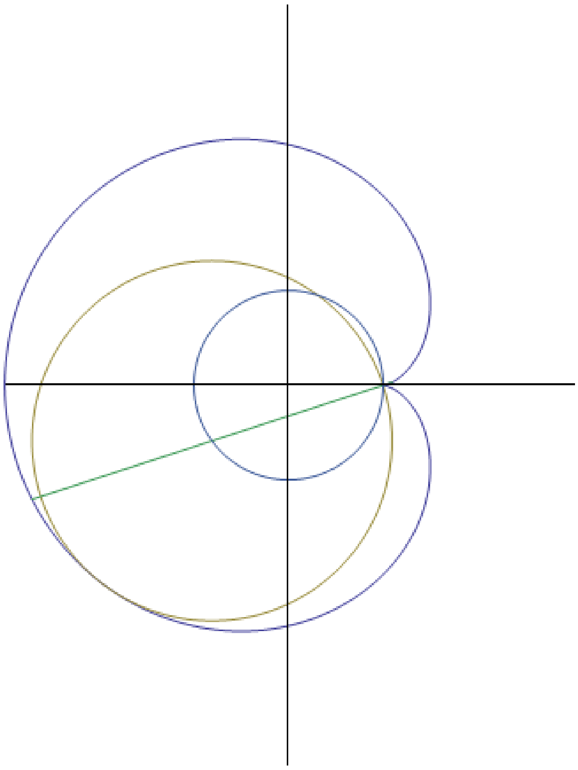

Figure 15: Couples cardioid/point envelopes and circle/line as ortho-cycloidals.

8.2 Couples 2-hypocycloid/nephroid envelope and cardioid/deltoid as ortho-cycloidals and rolling variable circle

Another example is given by the nephroid and the 2-hypocycloid which is the flat ellipse between the two cusps (a diameter of the fixed circle). The variable circle are centered on the fixed circle and tangent to x-axis, its radius is $r = \sin t$. The curve described by a point "angulary fixed" to the variable circle rolling without slipping on the diameter is a cardioid (as seen in parag. 3 above) and the curve described similarily by the circle rolling on the nephroid is a deltoid. So it is the couple of curves defined above as couples of orthogonal curves for rotation around the center O of the fixed circle. The cardioid constantly passes through the two cups of the nephroid, is tangent to the nephroid and its cusp stays on the 2-hypocycloid (the diameter). The deltoid passes through the cusps of the nephroid, is constantly tangent to the diameter and its 3 cusps move on the fixed nephroid. We have seen above similar property for a couple of moving cycloids.

8.3 Couples deltoid/3-epicycloid envelopes and nephroid/astroid as ortho-cycloidals and rolling variable circle

The next example is the couple of envelopes (deltoid and 3-epicycloids). One can find on web sites animations [see (8) (9) or (11)] as example of a couple of mirror curve and its caustic by reflection. For light rays coming from infinity the caustic by reflection of a deltoid is an astroid and if we change the direction of the rays this astroid turns inside the two cycloidale envelopes.

The astroid and the nephroid are the corresponding couple of ortho-cycloidals and can be generated by a variable circle rolling on these two curves generates a nephroid when rolling on the deltoid and an astroid when rolling on the 3-epicycloid. The nephroid has its cusps on the deltoid an is tangent to an arch of the 3-epicycloid. The astroid has its cusps on the 3-epicycloid and is tangent to an arch of the deltoid. If we change the point by the angle gap t2 on the variable circle all the positions can be obtained and permit to draw animations using only circles with fixed radius as a moving by rotation inside the two fixed cycloidals. An alternative animation can use an angulary fixed point on the variable circle.

Equations of the two envelope Epi- and Hypo-cycloids (outside (R) and inside (R) of the base circle) are :

$$X_1(t) = R\cos t + 2.r\sin[(R/(2r))t].\sin[t*(2r+R)/(2r)]$$

$$Y_1(t) = R\sin t - 2.r\sin[(R/(2r))t].\cos[t*(2r+R)/(2r)] \qquad (1)$$

$$X_2(t) = R\cos t + 2.r\sin[(R/(2r))t].\sin[t*(2r-R)/(2r)]$$
$$Y_2(t) = R\sin t - 2.r\sin[(R/(2r))t].\cos[t*(2r-R)/(2r)] \qquad (2)$$

The following curves are the couple the orthocycloidals turning inside the couple of two cycloidals with same cusps above. Equations of these locii of a point attached to the variable circle at position t_p rolling in opposite rotation repectively on the first couple of Epi-/Hypo-cycloidals envelopes above are:

$$X_3(t) = R\cos t + 2.r\sin[(R/(2r))(t)].\sin[(t-t_p)*R/(2r)]$$
$$Y_3(t) = R\sin t - 2.r\sin[(R/(2r))(t)].\cos[(t-t_p)*R/(2r)]$$
$$X_4(t) = R\cos t + 2.r\sin[(R/(2r))(t)].\sin[(t-t_p)*R/(2r)]$$
$$Y_4(t) = -R\sin t - 2.r\sin[(R/(2r))(t)].\cos[(t-t_p)*R/(2r)]$$

(Nota :The following paragraph needs to be improved)
These equations give means to draw all the curves in the figures of this paper. The element of arc length of the envelope k-epicycloidal (1) and tangent of the current tangent to the cycloidal are :

$$ds_1 = 2.(R+r)\sin[\frac{R}{2r}t]dt; \qquad \tan(\theta_1) = \tan[\frac{R+2r}{2r}t]$$

The element of arc length of the envelope k-Hypocycloidal (2) is :

$$ds_2 = 2.(R-r)\sin[\frac{R}{2r}t]dt; \qquad \tan(\theta_2) = \tan[\frac{2r-R}{2r}t]$$

For the rolling variable circle

$$R_{circle} = 2r\sin[(R/2r)t]$$

centered on base fixed circle the element of arc is given by

$$ds_{1-circle} = R_{circle}.d\theta_1 = (2r+R)\sin[(R/2r)t]dt$$

for rolling on cycloidal (Epi-envelope). The same size circle rolling in the opposite sense

$$ds_{2-circle} = R_{circle}.d\theta_2 = (2r-R)\sin[(R/2r)t]dt$$

will be used for rolling on cycloidal (Hypo-envelope). So we have the following formulas :

$$d\theta_1 = \frac{2r+R}{2r}dt; \qquad ds_{1-circle} = ds_1 - R.\sin[\frac{R}{2r}t]dt$$

$$d\theta_2 = \frac{2r - R}{2r} dt; \qquad ds_{2-circle} = -ds_2 + R.\sin[\frac{R}{2r} t] dt$$

These equations show that the resulting rolling is the composition of a the movement of the center of the variable circle and a rolling of this circle respectively on the envelope Epi-/Hypo-cycloidals above in the opposite direction of rotation.

On this example we see that the tangents at cusps are not necessary

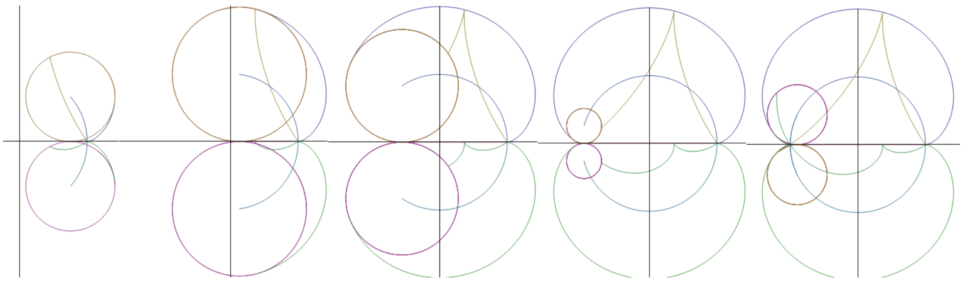

Figure 16: Progressive drawing of a the deltoid and cardioid. Orthogonal cardioid and deltoid can turn inside a nephroid. The variable circles rolling on fixed envelopes cycloidals once on the 2-Hypo and once on the nephroid generate respectively the cardioid and the deltoid.

orthogonal to the fixed curve (as for the ”roulettes”) because the variation of the radius modifies the trajectory. The rolling of the variable circle is not exactly similar to the rolling of a constant radius circle. The angle position on the variable circle varies continuously. And when the point passes through the circle with radius zero, the angle position is conserved (there is no angular discontinuity).

9 Final remarks.

This paper was inspired essentially by the two paper [R. Goormaghtigh (3) and J. Boyle (7)], the first author for its investigations on specific properties of envelopes of families of circles an the second for the new view on caustics, and specially the theorems 1 ”caustics” and 2 ”envelope theorem”. These envelopes of familes of circles certainly hide many other new properties waiting to be discovered.

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2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
There is an english adaptation.
Gregory's transformation : http://christophe.masurel.free.fr or
http://christophe.masurel.free.fr/#s9