DUPORCQ CURVES, STURMIAN SPIRALS, ROULETTES AND GLISSETTES - Part IX -

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Abstract

We present some properties of the Duporcq curves and Sturmian spirals from the point of view of Gregory's transformation. These two classes of plane curves are related to the conics and present interesting associated particularities.We use excentricity e of conics as the parmeter of form for the two classes (Duporcq and Sturm).

1 Curves of Duporcq - Sturmian spirals

In four papers of NAM, E. Duporcq, A. Manheim (1902), F. Balitrand (1914) and M. Egan (1919) gave the complete solution of the following problem of finding plane curves such that the element of arc length is e times (e= constant) the segment cut by the normals at the ends of this arc on a given line (the base line). The same problem but for the tangents (the arc of the curve is k times, k= constant, the segment cut by the tangents at the ends of this arc on a given line) leads to the Bouguer curves of pursuit as solutions (see Part VII). The two last authors above gave the name of Mannheim to the curves with vector radius proportional to the radius of curvature at the current point M. In fact these curves were mentioned by Sturm in 1857 in his course of analysis and we will keep this name sturmian used by various authors. The term of "Mannheim curve" is reserved to curves represented in a normal frame $f(x, y) = 0$ when $f(s, R_c) = 0$ for a given plane curve with x = s = curvilinear abcisse and $y = R_c$ = the radius curvature at the current point : this is the intrinsic equation of Cesaro. The sturmian spirals are the curves such that $R_c = e.\rho$ radius of curvature proportional to the vector radius in polar coordinates. A. Mannheim has proved that the roulette on a line of the pole of a Sturmian spiral is a Duporcq curve for directrice or base line (Δ).

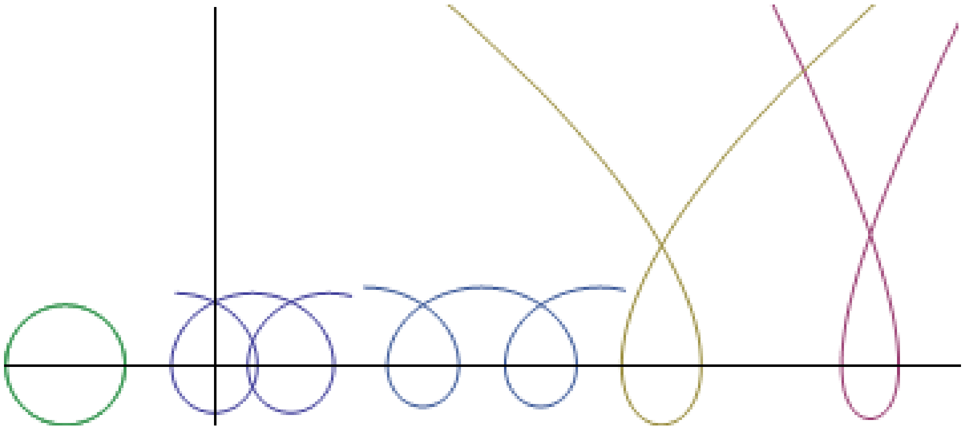

Figure 1: 3 Duporcq curves $e = 0 < e = 0.2, e = 0.4 < 1 = e, 1 < e$. The base line cuts orthogonally all curves.

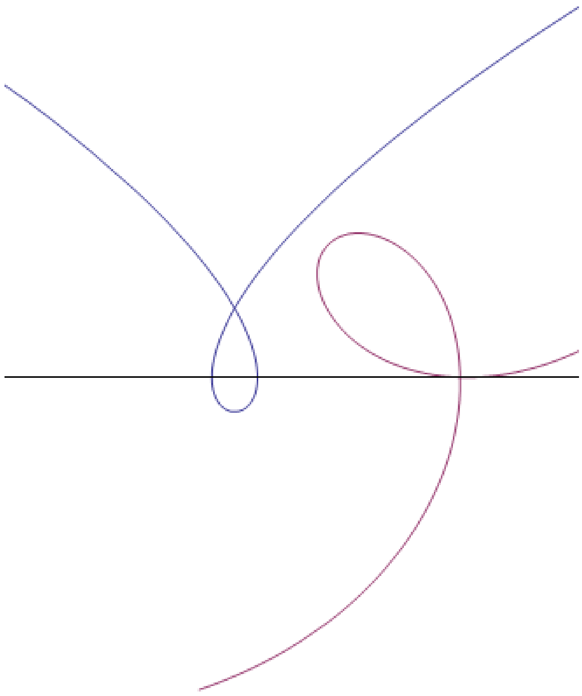

Figure 2: Roulette of Norwich Spiral is Tschirnhausen's cubic : parabolic case of Duporcq curve

2 Duporcq Curves generated by two motions : translation and central force conic

In a paper of 1902 E. Duporcq looked for plane curves such that the element of arc length is e times (e=constant ≥ 0) the segment cut on a given line

by the normals at the ends of this arc. M. Egan in1919 gave the complete solution with equations of all curves.
It is in fact possible to give the simple expressions for the Duporcq curves in cartesian parametric coordinates and the pedal of Sturmian spirals in polar coordinates (ρ, θ). This solution makes use of the composition of two motions: a Newton conical one (with e as the excentricity) which is itself also a composition of a rotation with constant speed u and uniform translation v and $e = u/v$. This Newtonian motion is composed with a translation w (uniform motion) orthogonal to the focal axis of the conic. The first motion is exactly the one a planet in orbit around the sun as the focus with constant areal velocity. It has equation (d constant) :

$$\rho = \frac{d}{v - u\cos\theta}$$

So the trajectory is a conic of equation in polar coordinates :

$$\rho = \frac{l}{1 - e\cos\theta}$$

It is an ellipse if $v > u$, an hyperbola if $v < u$ and a parabola if $u = v$.

If we combine the preceding motion with a translation w=e.v (uniform motion) in an orthogonal direction to the focal axis we obtain a new curve identified by M. Egan as a Duporcq curve. This permit him to obtain the cartesian equations of these curves in the 4 cases :
1- for the elliptic case if $0 \le e < 1$, if e=0 Duporcq cuve is the not modified circle
2- for lower branch of the hyperbolic case if $e > 1$
3- for upper branch of the hyperbolic case if $e > 1$
4- for parabola if e=1 : the Duporcq curve is the Tschirnhausen's cubic and the Sturmian spiral is the Norwich spiral (see Part IV).
For cases 2 and 3 we use M. Egan's names lower and upper but in our figures they are in the opposit position.

2.1 Elliptic Duporcq curves

Following M. Egan we define t the time, $l = b^2/a$ as parameter of the conic, $h = r^2.\frac{d\theta}{dt}$ areal velocity on the conic with focal axis along Oy. The Newtonian motion is defined by $u = h/l$ orthogonal to OP and $v = -he/l$. Then we give to the newtonian motion a constant speed : $u = h/l$ in the Ox direction. The resulting trajectory of P is a Duporcq curve described at constant speed $= h/l$.
The coordinates of P on the conic are in polar : r and θ, in cartesian (ς, ϖ).

M (x, y) is the current point of the Duporcq of curve and s is its arc length.
- for $0 \leq e < 1$, the parameter of the moving ellipse are :

$$l = b^2/a = a(1 - e^2)$$

$$\varsigma = -b \sin u \qquad \varpi = -ae + a \cos u$$

$$x = \frac{a^2}{b}(eu - \sin u) \qquad y = a(-e + \cos u)$$

$$s = \frac{a^2}{b}(u - e \sin u)$$

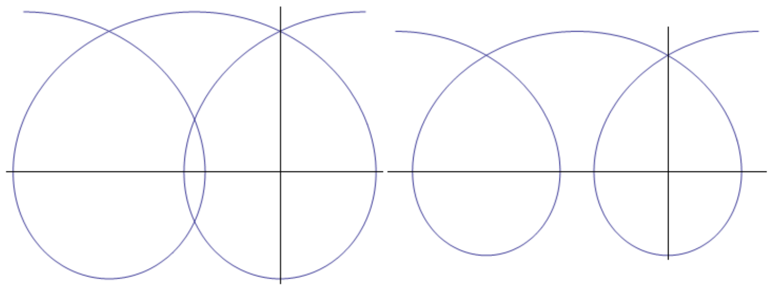

Figure 3: Duporcq curve for $(0 \leq e \leq 1)$ elliptic case

2.2 Hyperbolic Duporcq curves-A

- for $e > 1$, the parameter of the moving lower hyperbola are

$$l = b^2/a = a(e^2 - 1)$$

$$\varsigma = -b \sinh u \qquad \varpi = a(e - \cosh u)$$

$$x = \frac{a^2}{b}(\sinh u - eu) \qquad y = a(e - \cosh u)$$

$$s = \frac{a^2}{b}(e \sinh u - u)$$

2.3 Hyperbolic Duporcq curves-B

- for $0 \leq e < 1$, the parameter of the moving upper hyperbola are :

$$l = b^2/a = a(e^2 - 1)$$

$$\varsigma = -b \sinh u \qquad \varpi = a(e + \cosh u)$$

$$x = \frac{a^2}{b}(\sinh u + eu) \qquad y = a(e + \cosh u)$$

$$s = \frac{a^2}{b}(e \sinh u + u)$$

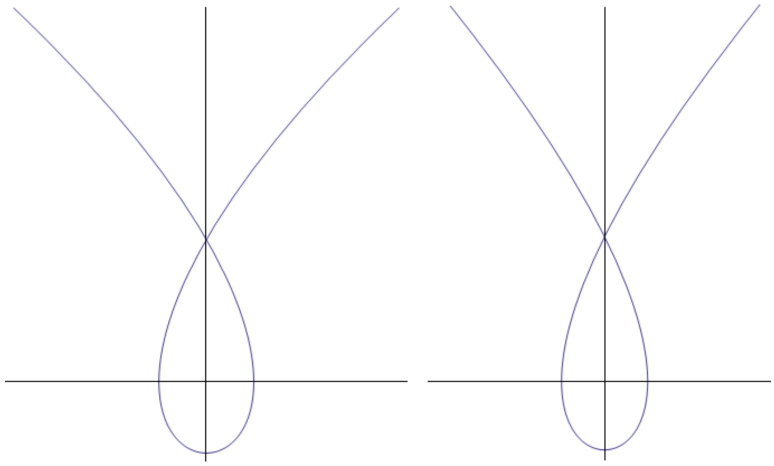

Figure 4: Duporcq curve TC for e=1 and for ($e \geq 1$) lower hyperbola

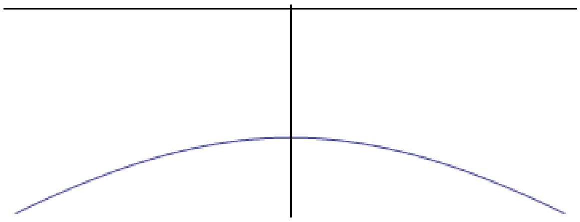

Figure 5: Duporcq curve 2-B ($e \geq 1$) upper hyperbola

2.4 Parabolic Duporcq curve

- for $e = 1$, the parameter of the moving parabola are

$$l = 6c \qquad \varsigma = -6cu \qquad \varpi = 3c(1-u^2)$$

$$x = c(u^3 - 3u) \qquad y = 3c(1-u^2)$$

$$s = c(3u + u^3)$$

This case is the couple Tschirnhausen's cubic as the roulette of Norwich spiral.

We see that all these curves have an arc length expressed by elementary functions. The elliptic case is a periodic function in x and period $T_x = 2n\pi a^2 e/b$. All have double point except hyperbolic-B case.

3 Wheels for the Duporcq Curves and base line

Since we have cartesian equations of the Duporcq Curves it is easy to calculate by Gregorys's transformation the polar equation of the wheels. We use inverse Gregory's transformation : $\rho = y$ and $\theta = \int_0^u \frac{dx}{y}$ so :

1- for elliptical case $0 \leq e < 1$:

$$\theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{a}{b} . \frac{(e - \cos u)}{(-e + \cos u)} du$$

$$\theta = \frac{-u}{\sqrt{1 - e^2}} \qquad y = a(-e + \cos u) = a[-e + \cos(\theta\sqrt{1 - e^2})]$$

$$\rho = a[\cos(\theta\sqrt{1 - e^2}) - e]$$

2- for hyperbolic case $e > 1$ lower branch of hyperbola :

$$\theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{a}{b} . \frac{(\cosh u - e)}{(e - \cosh u)} du$$

$$\theta = \frac{-u}{\sqrt{e^2 - 1}} \qquad y = a(e - \cosh u) = a[e - \cosh(\theta\sqrt{e^2 - 1})]$$

$$\rho = a[\cosh(\theta\sqrt{e^2 - 1}) - e]$$

3- for hyperbolic case $e > 1$ upper branch of hyperbola :

$$\theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{a}{b} . \frac{(\cosh u + e)}{(e + \cosh u)} du$$

$$\theta = \frac{-u}{\sqrt{e^2 - 1}} \qquad y = a(e + \cosh u) = a[e + \cosh(\theta\sqrt{e^2 - 1})]$$

$$\rho = a[e + \cosh(\theta\sqrt{e^2 - 1})]$$

4- for parabolic case $e = 1$:

$$\theta = \int_0^u \frac{dx}{y} = \int_0^u \frac{(3u^2 - 3u)}{3(1 - u^2)} du$$

$$\theta = -u \qquad y = 3c(1 - u^2) = 3c(1 - \theta^2)$$

$$\rho = 3c[1 - \theta^2]$$

All these are the wheels for a Duporcq curves as the grounds and the base line is the double normal. We know by Steiner-Habich theorem that these wheels are the pedals of the roulettes generating the Duporcq curves, so the four classes of curves above are the pedals of sturmian spirals.

4 Sturmian spirals

4.1 Some properties of Sturmian spirals - links with conics

The Sturmian spirals are the curves such that :

$$R_c = e.\rho$$

The radius of curvature is proportional ($e \geq 1$: parameter) to the vector radius in polar coordinates. Mannheim has shown that the roulette of the pole of a Sturmian spiral is a Duporcq curve. The special case e=1 is the spiral of Norwich the pole of which describes a TC when rolling on the double normal of the TC. Duporcq and Mannheim have showed in 1902 that Duporcq curves verify the formula :

$$\lambda^2 = a.\rho_c$$

where a is a parameter, ρ_c the radius of curvature and the value λ is the distance along the normal between the current point M of the curve and the point of intersection of the normal and the base line in the plane. The sturmian spiral verify a linear equation in ρ and p, the vector radius of the pedal at the current point :

$$e.\rho - e^2.p + A = 0$$

If V is the usual angle between Oy and the tangent at the current point, $y = \rho \sin V$ and so the preceeding formula can be written :

$$\rho = \frac{A/e}{-1 + e.\sin V}$$

So this is the equation of a conic in polar coordinates and angle V with focus at the origin of the frame.

4.2 Parametric equations of Sturmian spirals

F Balitrand (1914) and M. Egan (1919) noticed that sturmian spirals are involutes of cycloidal curves and this gives a geometric view of these spirals. Sturmian spirals are special involutes of epicycloids (elliptic case) or hypercycloids (hyperbolic cases) and the parabolic case is a special involute of the involute of the circle.
The curves found in preceeding section give a starting point to get the sturmian spiral equations since by Steiner-Habich theorem the antipedal of those curves are the spirals. The equations of these wheels supply the pedal equations of sturmian spirals. We know that angle V is conserved by pedal-antipedal transformations so the vector radius is :

$$r = p(\theta)/\sin V$$

and the angle V is given by usual polar formulas :

$$\tan V = \frac{p.d\theta}{dp}$$

Using the Euler normal (or pedal-)equation $x.\cos\theta + y\sin\theta = p(\theta)$ of the tangent to the curve theses formulas permit to derive the cartesian parametric equations of sturmian spirals. We give the arc length computed by formulas :

$$R_c = -[p(\theta) + p"(\theta)] = \frac{ds}{d\theta}$$

For the elliptic case :

$$x.\cos\theta + y.\sin\theta = \cos(\theta\sqrt{1-e^2}) - e$$

$$-x.\sin\theta + y.\cos\theta = -\sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2})$$

$$x = (\cos(\theta\sqrt{1-e^2}) - e).\cos\theta + \sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2}).\sin\theta$$

$$y = (\cos(\theta\sqrt{1-e^2}) - e).\sin\theta - \sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2}).\cos\theta$$

$$s = \frac{e}{\sqrt{1-e^2}}.\left[e.\sin[\theta\sqrt{1-e^2}] - \theta\sqrt{1-e^2}\right]$$

For the hyperbolic case (lower branch):

$$x.\cos\theta + y.\sin\theta = \cosh(\theta\sqrt{e^2-1}) - e$$

$$-x.\sin\theta + y.\cos\theta = \sqrt{e^2-1}\sinh(\theta\sqrt{e^2-1})$$

$$x = (\cosh(\theta\sqrt{e^2-1}) - e)\cos\theta - \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1}).\sin\theta$$

$$y = (\cosh(\theta\sqrt{e^2-1}) - e)\sin\theta + \sqrt{1-e^2}.\sinh(\theta\sqrt{e^2-1}).\cos\theta$$

$$s = \frac{e}{\sqrt{e^2-1}}.\left[e.\sinh[\theta\sqrt{e^2-1}] - \theta\sqrt{e^2-1}\right]$$

For the hyperbolic case (upper branch):

$$x.\cos\theta + y.\sin\theta = \cosh(\theta\sqrt{e^2-1}) + e$$

$$-x.\sin\theta + y.\cos\theta = \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1})$$

$$x = (\cosh(\theta\sqrt{e^2-1}) + e)\cos\theta - \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1}).\sin\theta$$

$$y = (\cosh(\theta\sqrt{e^2-1}) + e)\sin\theta + \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1}).\cos\theta$$

$$s = \frac{e}{\sqrt{e^2-1}}.\left[e.\sinh[\theta\sqrt{e^2-1}] + \theta\sqrt{e^2-1}\right]$$

And for the parbolic case (e=1):

$$x.\cos\theta + y.\sin\theta = 1 - \theta^2$$

$$-x.\sin\theta + y.\cos\theta = -2\theta$$
$$x = (1-\theta^2)\cos\theta + 2\theta.\sin\theta$$
$$y = (1-\theta^2)\sin\theta - 2\theta\cos\theta$$
$$s = \theta + \frac{\theta^3}{3}$$

Or in polar coordinates ($\tan u = \theta$):

$$\rho = \frac{1}{\cos^2 u} \qquad \theta = \tan u - 2u$$

The sturmian spiral is the Norwich spiral.

4.3 Parametric equations of the evolutes of Sturmian spirals

F Balitrand and M. Egan have shown that sturmian spirals are involutes the epicycloidal, hypercycloidal curves or evolute of the circle. The cycloidal curves are the anti-pedals of rhodoneas : $\rho = \cos k\theta$. It is an epicycloid if $k < 1$, a circle if $k = 1$ and an hypocycloid if $k > 1$.

The hypercycloids are anti-pedals of hyperbolic sine $\rho = \cosh k\theta$.

In the special parabolic case of sturmian spiral the Norwich spiral is the antipedal of $\rho = \theta^2 - 1$.

The first class has an intrinsic Cesaro $f(R,s)$ equation (a=constant) :

$$R^2 + k^2.s^2 = a^2$$
$$k^2.s^2 - R^2 = a^2$$

The last class studied by Cesaro is a sort of spiral turning around the pole and going to infinity. It has a cusp I and a symmetry axis OI.
These two classes of curves contains the evolutes of sturmian spirals since antipedals of rhodonea and of hyperbolic sine are respectively epicycloids with $k = \sqrt{1-e^2}$ $(e < 1)$ and hypercycloids with $k = \sqrt{e^2-1}$ $(e > 1)$. The parabolic case (e=1) corresponds to the spiral of Archimede for which the antipedal is the involute of the circle. The second involute passes through the middle of the segment IC (O center of the circle and I the cusp of the involute). It is the spiral of Norwich and the associated Duporcq curve is the TC.

4.3.1 Hypercycloids

The name of hypercycloids includes two sorts of cuves with intrinsic equations $k^2.s^2 - R^2 = a^2$ (curves H1) and $R^2 - k^2.s^2 = a^2$ (curves H2) studied by Euler who noticed that the two classes are evolutes one of the other. Curves H1 and H2 are spirals with a cusp for (H1) and without a cusp for (H2). The two classes H1/H2 of curves are associated to the positions of the hyperbola in the orthogonal frame. The evolutes of sturmian spirals are in H1 class and sturmian spiral are parallele to H2 curves. From the intrinsic equations of hypercycloids we can get cartesian paramatric equations (x,y) just like for epi/hypo-cycloidals curves. (Gomes Teixeira T.II pp 218-223):

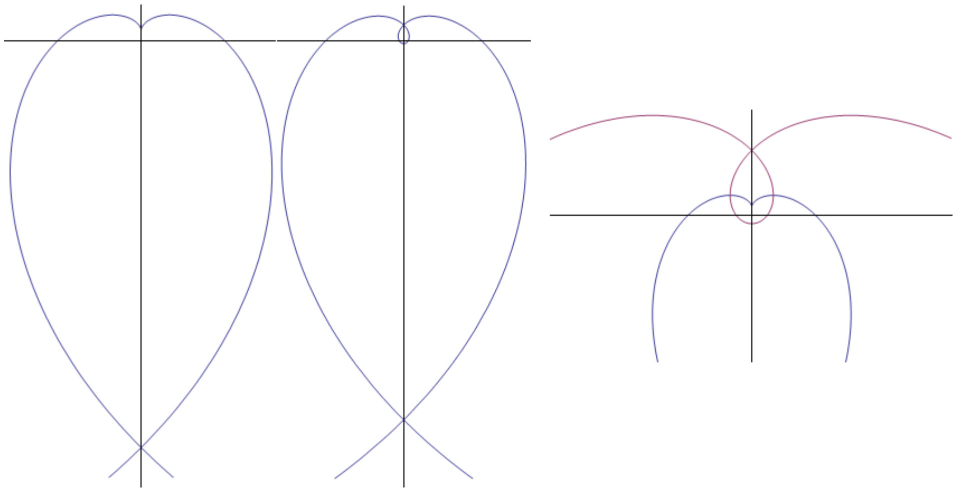

Figure 6: Hypercycloids with cusp (H1) : $k^2.s^2 - R^2 = a^2$, without cusp (H2): $R^2 - k^2.s^2 = a^2$

The class H1 : $k^2.s^2 - R^2 = a^2$

$$x = \frac{ak}{k^2+1}\left[\frac{1}{k}.\cosh(\phi/k).\cos\phi + \sinh(\phi/k).\sin\phi\right]$$

$$y = \frac{ak}{k^2+1}\left[\frac{1}{k}.\cosh(\phi/k).\sin\phi - \sinh(\phi/k).\cos\phi\right]$$

Equation of the pedal :

$$p = \frac{ak}{(k^2+1)}.\frac{1}{k}.\sinh(\theta/k)$$

The class H2 : $R^2 - k^2.s^2 = a^2$

$$x = \frac{ak}{k^2+1}\left[\frac{1}{k}.\sinh(\phi/k).\cos\phi + \cosh(\phi/k).\sin\phi\right]$$

$$y = \frac{ak}{k^2+1}\left[\frac{1}{k}.\sinh(\phi/k).\sin\phi - \cosh(\phi/k).\cos\phi\right]$$

Equation of the pedal :

$$p = \frac{ak}{(k^2+1)}.\frac{1}{k}.\cosh(\theta/k)$$

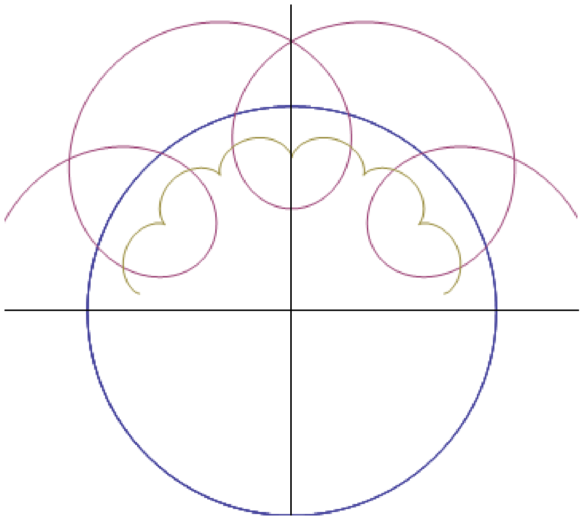

Figure 7: Sturmian curve Elliptic case and its evolute ($0 \leq e < 1$)

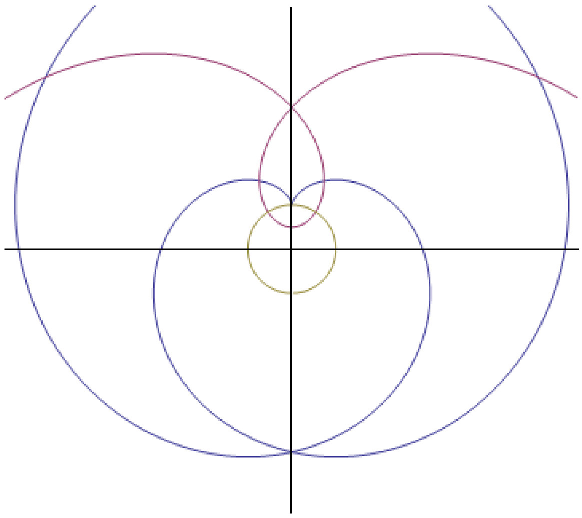

Figure 8: Sturmian Norwich Spiral and its evolute (involute of circle) ($e = 1$) parabolic case

4.3.2 Equations of the evolutes of sturmian spirals :

Since the pedal equations of these evolutes are :

$$p = \cos(\theta\sqrt{1 - e^2}) - e$$

$$p = \cosh(\theta\sqrt{1 - e^2}) \mp e$$

$$p = 1 - \theta^2$$

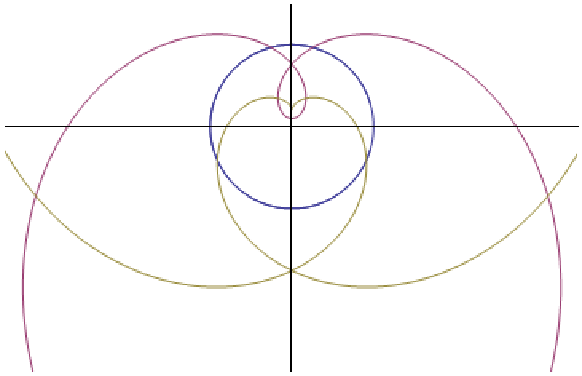

Figure 9: Sturmian curve Hyperbolic case and its evolute ($1 < e$)

We use the same method as above (Euler normal (or pedal-) equation) :
Elliptic case ($0 \leq e < 1$) :

$$-x.\sin\theta + y.\cos\theta = -\sqrt{1-e^2}\sin(\theta\sqrt{1-e^2})$$

$$-x.\cos\theta - y.\sin\theta = -(1-e^2)\cos(\theta\sqrt{1-e^2})$$

$$x = \sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2}).\sin\theta + (1-e^2).\cos(\theta\sqrt{1-e^2}).\cos\theta$$

$$y = -\sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2}).\cos\theta + (1-e^2).\cos(\theta\sqrt{1-e^2}).\sin\theta$$

Thes are equations of an epicycloid.
Elliptic sturmian spirals are involutes of epicycloids.
Hyperbolic lower and upper cases ($e > 1$):

$$-x.\sin\theta + y.\cos\theta = \sqrt{e^2-1}\sinh(\theta\sqrt{e^2-1})$$

$$-x.\cos\theta - y.\sin\theta = (e^2-1)\cosh(\theta\sqrt{e^2-1})$$

$$x = -\sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1}).\sin\theta - (e^2-1)\cosh(\theta\sqrt{e^2-1}).\cos\theta$$

$$y = \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1}).\cos\theta - (e^2-1)\cosh(\theta\sqrt{e^2-1}).\sin\theta$$

Thes are equations of an hypercycloid.
Hyperbolic sturmian spirals are involutes of hypercycloids.
Parabolic case (e=1) :

$$-x.\sin\theta + y.\cos\theta = -2\theta$$

$$-x.\cos\theta - y.\sin\theta = -2$$

$$x = 2\theta.\sin\theta + 2\cos\theta$$

$$y = -2\theta.\cos\theta + 2\sin\theta$$

Or in polar coordinates ($\tan u = \theta$):

$$\rho = \frac{2}{\cos u} \qquad \theta = \tan u - u$$

This evolute is, as expected, an involute of the circle.
Parabolic sturmian spiral is an involute of the involute of the circle.

5 Conics and sturmian spirals

We have already seen that a motion on a conic and a translation can be used to define Duporcq curves, roulette on a base line of sturmian spirals.

Glissettes - Roulettes For a curve (the couple [(C),O]) in polar coordinates is gliding constantly tangent at a the same point I on a fixed base line then the trajectory of the point O defines the glissette.
If a curve (the couple [(C),O]) in polar coordinates is rolling without slipping on the same fixed base line then the trajectory of the point O defines the roulette.
If we take V as the usual angle between vector ray ρ and the oriented tangent in polar coordinates (ρ, θ) then the equation of the glissette is :

$$(\rho, V)$$

The glissette of (C) is also the roulette of the evolute of (C) on the line orthogonal to the base line at the fixed point on the base line. The roulette on xx' of the couple [(C), O] curve in polar coordinates (ρ, θ) :

$$y = \rho \sin \theta \text{ and } x = \int_{\theta 0}^{\theta 1} ds_w - \rho \cos V$$

A roulette always cuts orthogonally the base line since the center of rotation is on this line.

The roulette (R) of (C) is a deformation of a glissette (G) of the same curve (C) w.r.t. the same base line by a varying translation equal to the length s of (C) and parallele to the base-line. For a closed curve (C) with smooth tangent $\rho = f(\theta)$ there is an associated glissette (G) $\rho = f(V)$ and a roulette (R) that have interacting properties since they are generated by the same initial curve (C).

5.1 Glissettes of Sturmian spirals.

We use the fact that angle V for the pedal in polar coordinates is equal to the one of initial curve. We have computed the equations of pedals $p = p(\theta)$

of sturmian spirals so angle V is given by (ρ for antipedal) :

$$\tan V = \frac{p.d\theta}{dp}$$

$$\rho = \frac{p}{\sin V}$$

$$x = \rho.\cos V = p/\tan V \quad and \quad y = p$$

we apply this to :
Glissette of elliptic sturmian spiral is an ellipse :

$$p = \cos(\theta\sqrt{1-e^2}) - e$$

$$\tan V = \frac{p.d\theta}{dp} = \frac{\cos(\theta\sqrt{1-e^2}) - e}{-\sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2})}$$

$$x = \sqrt{1-e^2}.\sin(\theta\sqrt{1-e^2})$$

$$y = \cos(\theta\sqrt{1-e^2}) - e$$

Glissette of (lower) hyperbolic sturmian spiral is a branch of hyperbola:

$$p = \cosh(\theta\sqrt{e^2-1}) - e$$

$$\tan V = \frac{p.d\theta}{dp} = \frac{\cosh(\theta\sqrt{e^2-1}) - e}{\sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1})}$$

$$x = \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1})$$

$$y = \cosh(\theta\sqrt{e^2-1}) - e$$

Glissette of (upper) hyperbolic sturmian spiral is a branch of hyperbola:

$$p = \cosh(\theta\sqrt{e^2-1}) + e$$

$$\tan V = \frac{p.d\theta}{dp} = \frac{\cosh(\theta\sqrt{1-e^2}) + e}{\sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1})}$$

$$x = \sqrt{e^2-1}.\sinh(\theta\sqrt{e^2-1})$$

$$y = \cosh(\theta\sqrt{e^2-1}) + e$$

Glissette of parabolic sturmian spiral (or Norwich) e=1is a parabola :

$$p = 1 - \theta^2$$

$$\tan V = \frac{p.d\theta}{dp} = \frac{1-\theta^2}{-2.\theta}$$

$$x = -2.\theta$$

$$y = 1 - \theta^2$$

with new variable u ($\theta = \tan u$) :

$$\tan V = \tan(2u - \pi/2)$$

$$x = \rho \cos V = \frac{1}{\cos(V/2)^2} \quad \text{or} \quad y = 1 - x^2/4$$

This is a parabola, pole at the focus.
So we have the property : the glissettes of sturmian spirals are conics with the focus at the point on the fixed tangent. The case of the circle is special: Sturmian spiral is reduced to a point at distance R from the pole ($\theta = 0$). The pedal is a circle $\rho = \cos\theta$ antipedal of the point and the glissette is a circle (e=0). We note that the translation by s along the base line x'x applied to the points of the conic/glissette is equivalent to the constraint imposed by constant areal velocity on the conic motion mentioned by M. Egan for the cinematic generation of Duporcq curves. The two conics (Glissette and Newtonian conic) are identical.

6 Arc length of Duporcq of Curves - Sturmian spirals and their pedals

These curves depend on one parmeter e and a curve in one family is associated uniquely to a curve in the other with same e. The pedals of the Sturmian spirals w.r.t. the pole are wheels for the Duporcq curves if we take the double or multiple normal (the directrix of the curves) as axis x'x, consequence of the theorem of Steiner - Habich (see Part I). F. Balitrand has shown that arc length elements verify :

$$ds_d = ds_p = e.ds_s$$

ds_d, ds_p and ds_s are respectively the corresponding arc elements of Duporcq, pedals of sturmian spirals and sturmian spirals. This follows also from the fact that the wheel (pedal of the sturmian spiral) has the same arc length as the corresponding ground (the Duporcq curve).

7 Tschirnhausen's cubic as the conclusion

The Tschirnhausen's cubic is a Duporcq curve for e=1 and a Bouguer curve of pursuit for k=1/2. So two normals to the TC at the ends of an arc cuts on the double normal (2^{nd}base) an equal lenghth as s and the two tangents at the ends of the same arc cuts half the length of s on the tangent at the summit S (see part IV).

Annexe I :

Roulette, Pedal and Gregory's transformations.

Parametric equations of the roulette (x, y) of the pedal (ρ, θ) and Gregory transformations (GT, GT^{-1})

Roulette : the formulas to determine the cartesian parametric equation (x, y) of the roulette on the line xx' of a curve given in polar coordinates (ρ, θ) :

$$y = \rho \sin(\theta) \text{ and } x = \int_{\theta 0}^{\theta 1} ds_w - \rho \cos(V)$$

Pedal : the formula to obtain the parametric polar equation of the pedal of the polar curve (ρ_0, θ_0) of a curve given in polar coordinates $(\rho_1, \theta_1$ is :

$$\rho_1 = \rho_0 \sin(V) \text{ and } \theta_1 = \theta_0 - (V - \pi/2)$$

The equation of n^{th} pedal is $(n \in \mathbb{Z})$:

$$\rho_n = \rho_0 \sin^n(V) \text{ and } \theta_n = \theta_0 - n(V - \pi/2)$$

Gregory : The formulas associated to direct (GT) and inverse (GT^{-1}) Gregory's transformation from the wheel to the ground, direct Gregory's Transform is :

$$y = \rho \qquad \text{and} \qquad x = \int \rho.d\theta$$

in the opposite way GT^{-1} from the ground to the wheel :

$$\rho = y \quad \text{and} \quad \theta = \int \frac{dx}{y}$$

$$\tan V = \frac{\rho.d\theta}{d\rho} = \frac{dx}{dy}$$

The GT^{-1} is defined in the whole euclidean plane except on the line $y = 0$ for which the angle of the wheel is not defined. This line is called the "base-line"and is the dual of the pole. A translation in the ground-plane corresponds to a rotation in the wheel-plane.

This article is the IX^{th} part on a total of 9 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds

Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english version.

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Nouvelles Annales de Mathematiques (1842-1927) NAM (Archives Gallica online)
E. Duporcq NAM 1902 p 181,
A. Mannheim : NAM 1902 p337 et p481,
F. Balitrand : NAM 1914 p97 and M Egan : NAM 1919 p241-248.