

GREGORY'S TRANSFORMATION EULER/SERRET CURVES WITH SAME ARC LENGTH AS THE CIRCLE

Part - II

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Abstract

We use the Gregory's transformation to study a problem from Euler: to find algebraic curves with the same arc length as the circle. Serret, in the middle of 19th century found, using an algebraic calculation, an infinite serie of algebraic curves. Gregory's formulas give a solution and a geometric interpretation of the results of Euler and Serret. This could be used to find curves with same arc length as the ellipse or the lemniscate.

1 Euler's problem and Gregory's transformation

In a paper of 1781 (published in latin about 1830 : "De curvis algebraicis, quarum omnes arcus perarcus circulares metiri liceat" Euler searched for curves with an arc length equal to the arc of a circle. He gave the expression of ρ in the form : $d + R \cos(\theta)$.

It seems he used a geometrical argument similar to the reverse Gregory's transformation (GT^{-1}), given the ground to find the corresponding wheel (see Annexe I).

Indeed, applying (GT^{-1}) to the problem consists in looking for a solution to a not conventional question to find a wheel that rolls on a ground of circular form in the fixed plane (x, y) . Euler gave a class of solutions and studied the first case (Fig.1) which looks like a lemniscate with two unequal loops.

2 The general algebraic solutions of J.A. Serret

Serret published in 1852 a paper in the Journal de l'Ecole Polytechnique on the Euler problem of finding curves with same arc length as the circle and used

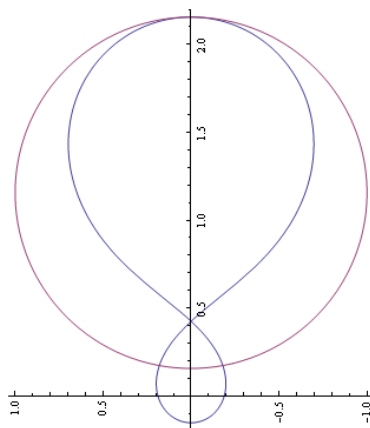


Figure 1: 1st curve of Euler length 2π

similar algebraic methods. He found an infinite class of such curves by means involving elementary arcs length of circle and integration of rational expressions equal to this arc. Euler's solutions are a subclass of Serret's.

3 A solution to Euler's problem by Gregory's transformation :

The Gregory's transformation is a geometric tool which associates two curves : a ground in cartesian and a wheel in polar coordinates : it defines a plane motion by rolling so that the pole of the wheel runs along the $x'x$ axis when the wheel rolls on the ground.

We apply the reverse Gregory's Transformation (GT^{-1}) which gives the equations of the wheel when we know the ones of the ground: a circle of radius one with center at a distance d from the base-line. And we look for the corresponding wheel.

We suppose that the circle (radius = 1) doesn't intersect with the base-line ($d > 1$) so to avoid singularities and that the wheel is smooth everywhere. By definition of the GT^{-1} the wheel is a solution of the problem because the two curves (ground-circle and wheel) have the same arc length by rolling one on the other without slipping. We choose the angle between vertical axis y upward and the radius of the circle at the current point as the parameter of the circle : $t=0$ at the top and $t = \pm\pi$ at the bottom of the circle and is counted from the vertical line.

The parameter d (distance from center of the circle to the base-line) is the only value which decides the type of resulting curve.

Equations of the ground :

$$x = \sin t \quad y = d + \cos t$$

Computation of the polar parametric equations of the wheel:

$$\rho = y = d + \cos t \quad \theta = \int \frac{\cos t}{d + \cos t} dt$$

The polar angle θ is given by an elementary arc tangent function we find :

$$\rho = d + \cos t \quad \theta = t - \frac{2d}{\sqrt{d^2 - 1}} \arctan \left[\sqrt{\frac{d-1}{d+1}} \tan \frac{t}{2} \right]$$

These equations give part of the wheel corresponding to one turn of the circle/ground (from $t = -\pi$ to $t = +\pi$) that is repeated. Generally the wheel doesn't close (there is a gap at each turn of the circle-ground), covers a ring around the pole and is not algebraic.

The algebraic solutions found by Euler were closed. This requires that the an-

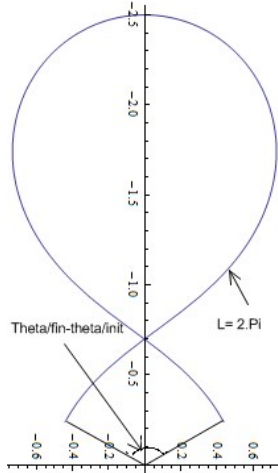


Figure 2: wheel (not closed)

gle (measured from the pole) between the initial point ($t = -\pi$) and final point ($t = +\pi$) of the wheel is commensurable to 2π . Let's suppose this proportion is k/n with $k, n \in \mathbb{N}$ the condition becomes $\theta_{fin} - \theta_{init} = 2\pi(k/n)$. With the above formula for θ the condition is :

$$\pm 2\pi \frac{k}{n} = \pi - (-\pi) - \frac{2\pi d}{\sqrt{d^2 - 1}}$$

$$1 + \frac{k}{n} = \frac{d}{\sqrt{d^2 - 1}}$$

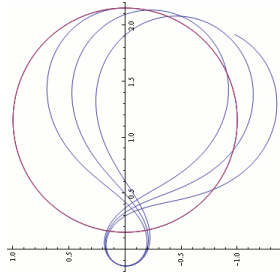


Figure 3: wheel (not closed)

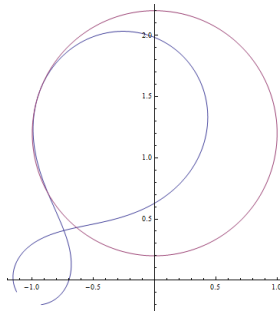


Figure 4: wheel (not closed)

So the values of d for algebraic (closed) curves are :

$$d = \frac{n + k}{\sqrt{k^2 + 2nk}}$$

$$\rho = \frac{n + k}{\sqrt{k^2 + 2nk}} + \cos(t)$$

$$\theta = t - 2 \cdot \left[\frac{n + k}{n} \right] \cdot \arctan \left[\frac{(n + k) - \sqrt{(k^2 + 2nk)}}{n} \cdot \tan \frac{t}{2} \right]$$

(added in 27/07/2017) : "To make the connection with recent Part XX^{th} , it is possible to use a change of notation to connect those formulas to the notations of F. Morley where b takes place of d and a replaces 1 (radius of the circle) to get simpler formula : $n + k \mapsto p$ and $k \mapsto q$ so

$$d = b = \frac{p}{\sqrt{p^2 - q^2}} \quad \rho = \frac{p}{\sqrt{p^2 - q^2}} + \cos(t)$$

$$\theta = t - 2 \cdot \left[\frac{p}{p - q} \right] \cdot \arctan \left[\frac{p - \sqrt{p^2 - q^2}}{p - q} \cdot \tan \frac{t}{2} \right]$$

and

$$\frac{p}{q} = \frac{b}{\sqrt{b^2 - a^2}} \text{ " end addendum(27/7/17).}$$

4 Two classes of curves among the solutions :

The general solution leads to an infinite number of curves so we will limit the purpose to two simple series of solutions for $k=1$ or $n=1$.

The curve ($n = k = 1$) $d = \frac{2}{\sqrt{3}}$ which is the first curve found by Euler is called the fundamental : two loops a big one and a small one around the pole.

The first class is a decreasing serie of closed curves when the base line approaches the tangent with increasing integer number of small inside loops around the pole. This serie has a limit curve which is the inverted of Norwich spiral (w.r.t. the pole) with asymptotic point at this pole. Each of these curves has a length of $2.\pi$.

The second class is an increasing serie of closed curves when the base-line goes away to ∞ and these curves present an integer number L of big loops and the global length is : $L.2.\pi$. These curves are contained in a space between two concentric circles $R = d + 1$ and $R = d - 1$.

And there is also an infinite number of intermediate cases when n and p take all values in \mathbb{N} .

4.1 The decreasing serie base line approaches the tangent to the circle at the lowest point :

Decreasing serie $n = 1, k \in \mathbb{N}^*$:

$$d = \frac{1 + k}{\sqrt{k^2 + 2.k}}$$

$$\rho = \frac{1 + k}{\sqrt{k^2 + 2.k}} + \cos(t)$$

$$\theta = t - 2.(1 + k). \arctan \left[[1 + k - \sqrt{k^2 + 2.k}]. \tan \frac{t}{2} \right]$$

For $k = 1, d = \frac{2}{\sqrt{3}}$ the curve is the first found by Euler :

$$\rho = \frac{2}{\sqrt{3}} + \cos t \quad \theta = t - 4. \arctan \left[(2 - \sqrt{3}). \tan \frac{t}{2} \right]$$

For $k = 2, d = \frac{3}{2\sqrt{2}}$ the curve has a second loop inside the small loop :

$$\rho = \frac{3}{2.\sqrt{2}} + \cos t \quad \theta = t - 6. \arctan \left[(3 - 2.\sqrt{2}). \tan \frac{t}{2} \right]$$

Equation for $k=3$: $d = \frac{4}{\sqrt{15}}$ the curve has two loops inside the small loop :

$$\rho = \frac{4}{\sqrt{15}} + \cos t \quad \theta = t - 8. \arctan \left[(4 - \sqrt{15}). \tan \frac{t}{2} \right]$$

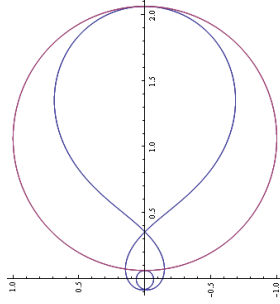


Figure 5: 2nd curve of Euler ($d = \frac{3}{2\sqrt{2}}$) length : 2π

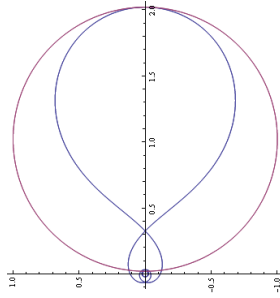


Figure 6: 3rd curve of Euler ($d = \frac{4}{\sqrt{15}}$) length : 2π

This serie has a limit curve with an infinite of turns around the asymptotic pole O, the base line is tangent to the circle ($d=1$) and its equations are :

$$\rho = \tan t - 2t \quad \theta = 2 \cos^2 t$$

This curve is the transformed by inversion spiral of Norwich (Sylvester 1868) and is not algebraic.

4.2 The increasing serie for d going to infinity :

Increasing serie $k=1$, $n \in \mathbb{N}$:

$$d(n) = \frac{n+1}{\sqrt{2n+1}}$$

$$\rho = \frac{n+1}{\sqrt{1+2n}} + \cos t$$

$$\theta = t - 2 \cdot \left[\frac{n+1}{n} \right] \arctan \left[\frac{(n+1) - \sqrt{1+2n}}{n} \cdot \tan \frac{t}{2} \right]$$

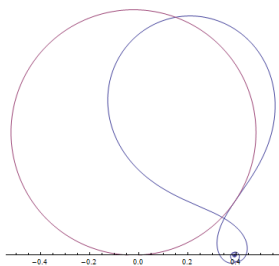


Figure 7: Limit case wheel for circle ground when base-line is tangent to the circle.

Equation for n=1 : it is the fondamentale first curve studied by L. Euler(fig.1).

Equation for n=2 : $d = \frac{3}{\sqrt{5}}$

$$\rho = \frac{3}{\sqrt{5}} + \cos t \quad \theta = t - 3 \arctan \left[\frac{(3 - \sqrt{5})}{2} \cdot \tan \frac{t}{2} \right]$$

Equation for n=3 : $d = \frac{4}{\sqrt{7}}$

$$\rho = \frac{4}{\sqrt{7}} + \cos t \quad \theta = t - \frac{8}{3} \arctan \left[\frac{(4 - \sqrt{7})}{3} \cdot \tan \frac{t}{2} \right]$$

Equation for n=4 : $d = \frac{5}{3}$

$$\rho = \frac{5}{3} + \cos t \quad \theta = t - \frac{5}{2} \arctan \left[\frac{2}{3} \cdot \tan \frac{t}{2} \right]$$

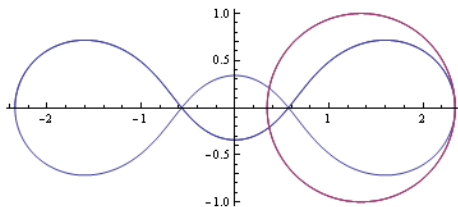


Figure 8: 2nd curve ($d = \frac{3}{\sqrt{5}}$) the total length is 4π

4.3 Analogy with the regular polygons :

If the distance d from center of the circle to the line is increasing we obtain many intermediate cases and the situation is exactly the same as for polygons inscribed in a circle if we increase the repeated initial basic angle at the center. When this angle is not commensurable with the value 2π the polygone

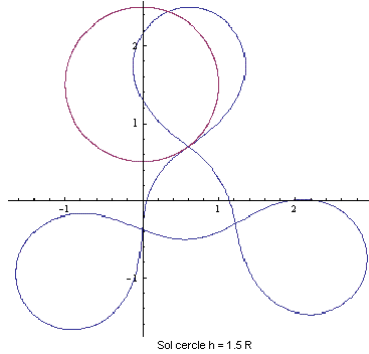


Figure 9: 3rd curve ($d = \frac{4}{\sqrt{7}}$) the total length is 6π

doesn't close and fills the interior of the space between two circles (same as for trochoids if the two radii are not commensurable). The corresponding angle for Euler problem is the angular distance measured from the pole between two points after a turn of the circle-ground.

As in regular polygons (or trochoids generated by two circles rolling one on another), there are only two possibilities for the couple ground-wheel :

- 1- the value of this angle $\theta_{fin} - \theta_{init}$ is incommensurable to 2π then the wheel never returns to initial position and the curve cover all the space between two concentric circles and its equation is not algebraic.
- 2- the value of this angle is equal to $(k/n).2\pi$ with $k, n \in \mathbb{N}$ then the wheel is closed after an integer number of turns and is algebraic. This is the case if d has the following algebraic value depending on two integer parameters n and k and give the Euler general solutions :

$$d = \frac{n + k}{\sqrt{k^2 + 2nk}}$$

Remark : We have admitted the possibility that the global length of the curve has a length p times 2π ($p \in \mathbb{N}$) otherwise the only acceptable class of solutions is the one approaching the tangent (first serie above) corresponds to the wheels associated to one complete cycle of the circle-ground.

4.4 Conclusion

Serret presented the algebraic solutions by equation $\cos(\theta) = f(\rho)$ or $\sin(\theta) = g(\rho)$ where f, g are rational functions in ρ . The elimination of t between the two equations (see 3 above) must be possible for the serie of closed wheels. The few curves that Serret cited in his work are those that can be found with Gregory's transformation that gives an intuitive understanding of the generation

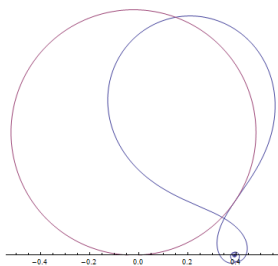


Figure 10: A wheel for circle-ground (base-line : tangent to the circle). It presents an asymptotic point at the pole and is the inverted of Norwich spiral w.r.t. the pole.

by rolling of these closed curves (closing is a necessary condition for the curve to be algebraic).

Curves of Serret corresponding to $d = 2/\sqrt{3}$ and $d = 3/(2\sqrt{2})$ are :

$$3\rho^2 \cdot \sqrt{3} \cdot \cos \theta = \rho^3 + 6\rho - 2$$

$$8\rho^3 \cdot \cos \theta = \rho^4 + 14\rho^2 - 8\rho + 1$$

But the number of algebraic solutions of the problem could be much bigger than we imagine because the Euler problem is widely undetermined.

Indeed an algebraic equation in cartesian (x,y) - $g(x,y)$ - is a geometric relation between the distances from the current point of the curve and two orthogonal lines (orthonormal axis). So the algebraic curve is a also geometric object.

In this paper we used a line (the base-line) to define a curve rolling on the fixed algebraic circle but we could use another algebraic curve (as a circle) to be the fixed curve where the pole must run when the wheel is rolling. The structure of rolling curves in the plane has not revealed all secrets.

This article is the 2^{nd} on plane curves.

Part I : Gregory's transformation.

Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.

Part III : A generalization of sinusoidal spirals and Ribaucour curves

Part IV: Tschirnhausen's cubic.

Part V : Closed wheels and periodic grounds

Part VI : Catalan's curve.

Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β -curves.

Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.

Part IX : Curves of Duporcq - Sturmian spirals.

Part X : Intrinsically defined plane curves, periodicity and Gregory's transformation.

Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
 Part XII : Caustics by reflection, curves of direction, rational arc length.
 Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
 Part XIV : Variable epicycles, orthogonal cycloidal trajectories, envelopes of variable circles.
 Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
 Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
 Part XVII : Cesaro's curves - A generalization of cycloidals.
 Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
 Part XIX : Tangential generation, curves as envelopes of lines or circles, arcuïdes, causticoïdes.
 Part XX : Tangential dual of Steiner Habicht theorem, Circular tractrices, newtonian catenaries, circles as roulettes of a curve on a line.

There are also two papers I have published in french :

1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (n 15 Fevrier 1983)

2- Une generalisation de la roue - Bulletin de l'APMEP (n 364 juin1988).

There is an in english translation.

References :

- Traite des courbes speciales remarquables - Gomez Teixeira 1913 (Chelsea 1971)
- Courbes geometriques remarquables - H.Brocard-E.Lemoine (Blanchard 1967)
- Euler : De curvis algebricis, quarum omnes arcus per circulares metiri liceat (1781).
- J.A. Serret : Memoire sur les courbes dont les arcs s'expriment par des arcs de cercle - Journal de l'Ecole Polytechnique (1853).
- Prasolov and Solovyev : Elliptic functions and elliptic integrals.(1997)

Annexe I

Roulette, Pedal and Gregory's transformations.

Parametric equations of the roulette (x, y) of the pedal (ρ, θ) and Gregory transformations (GT, GT^{-1})

Roulette : the formulas to determine the cartesian parametric equation (x, y) of the roulette on the line xx' of a curve given in polar coordinates (ρ, θ) :

$$y = \rho \sin(\theta) \text{ and } x = \int_{\theta_0}^{\theta_1} ds_w - \rho \cos(V)$$

Pedal : the formula to obtain the parametric polar equation of the pedal of the

polar curve (ρ_0, θ_0) of a curve given in polar coordinates (ρ_1, θ_1) is :

$$\rho_1 = \rho_0 \sin(V) \text{ and } \theta_1 = \theta_0 - (V - \pi/2)$$

The equation of n^{th} pedal is $(n \in \mathbb{Z})$:

$$\rho_n = \rho_0 \sin^n(V) \text{ and } \theta_n = \theta_0 - n(V - \pi/2)$$

Gregory : The formulas associated to direct (GT) and inverse (GT^{-1}) Gregory's transformation from the wheel to the ground, direct Gregory's Transform is :

$$y = \rho \quad \text{and} \quad x = \int \rho \cdot d\theta$$

in the opposite way GT^{-1} from the ground to the wheel :

$$\rho = y \quad \text{and} \quad \theta = \int \frac{dx}{y}$$

$$\tan V = \frac{\rho \cdot d\theta}{d\rho} = \frac{dx}{dy}$$

The GT^{-1} is defined in the whole euclidean plane except on the line $y = 0$ for which the angle of the wheel is not defined. This line is called the "baseline" and is the dual of the pole. A translation in the ground-plane correspond to a rotation in the wheel-plane.

Annexe II

Some examples of multiloops wheels (the 2nd serie and intermediate cases) :

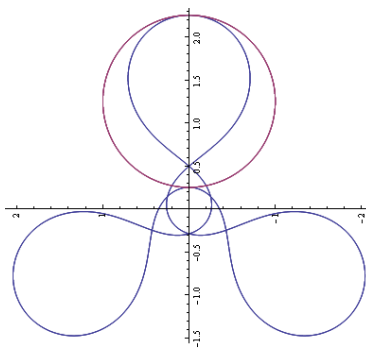


Figure 11: curve 3-2 loops the total length is 6π

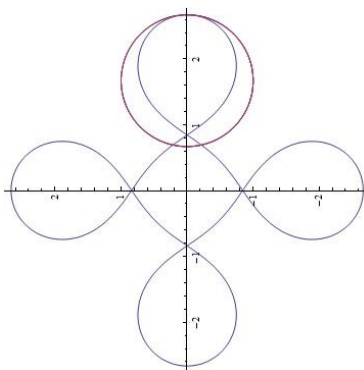


Figure 12: curve 4-1 loops the total length is 8π

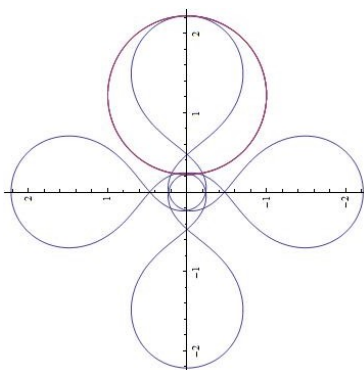


Figure 13: curve 4-3 loops the total length is 8π

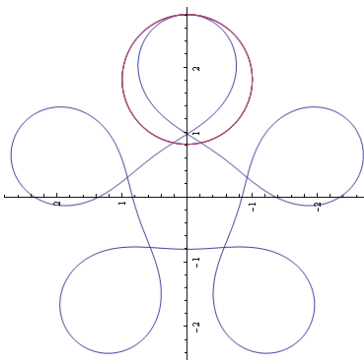


Figure 14: curve 5-1 loops the total length is 10π

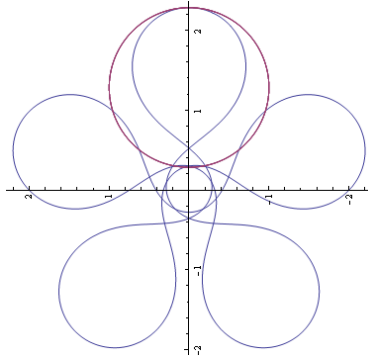


Figure 15: curve 5-3 loops the total length is 10π

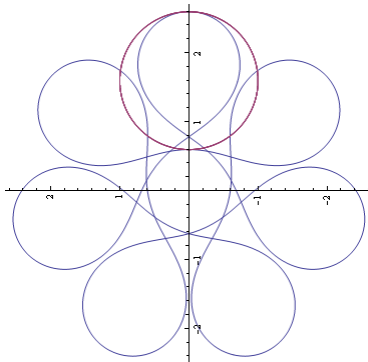


Figure 16: curve 7-2 loops the total length is 14π