GREGORY'S TRANSFORMATION
Part - I

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Abstract

James Gregory in his book "Geometria pars universalis -1668" presented a general transformation in the plane which associates a curve in polar coordinates (ρ, θ) to a curve in cartesian orthonormal coordinates (x, y). We examine geometric properties of these corresponding couples of plane curves and use the terms ground (x,y) and wheel (ρ, θ) to name the couple of associated objects defined by Gregory's transformation.

1 Polar coordinates, cartesian coordinates and the Gregory's transformation.

The rolling motion of a curve in a mobile plane on another curve in a fixed plane took part in 17^{th} century to the developpement of new methods on curves like the roulette or cycloid : the track of a point of a circle rolling on a straight line. But this subject has declined since that time.
In this paper we recall some properties of curves rolling in the plane and examine the specific case of couple of curves linked by the Gregory's transformations.
We use the parametric equation (x,y) or $(\rho,\ \theta)$ as functions of a single parameter to define the curves.
Gregory's transformation associates two plane curves, one in polar coordinates (ρ, θ) and the other in cartesian orthonormal (x,y)-frame, and is defined in the following way :

$$y = \rho \qquad \text{and} \qquad x = \int_{\theta o}^{\theta} \rho . d\theta \quad \text{(1) Direct Gregory's Transformation)}$$

or in the opposite way :

$$\rho = y \qquad \text{and } \theta = \int_{xo}^{x} \frac{dx}{y} \qquad y \neq 0 \text{ (2) Gregory's Transformation}^{-1}\text{)}$$

$$\text{Define :} \qquad \tan V = \frac{\rho . d\theta}{d\rho} = \frac{dx}{dy} \text{ (3)}$$

Remark 1: For the wheel V is the usual angle between ρ and the oriented tangent line. For the ground the angle V is defined between y'y and the oriented tangent. It is not the usual angle γ between axe xx' and the oriented tangent.
Remark 2: We note that GT^{-1} is defined in the plane except on the line $y = 0$. This line is called the "base-line" or "critical line".
Remark 3 : The rolling move is supposed to be without slipping so $s_w = s_r$.

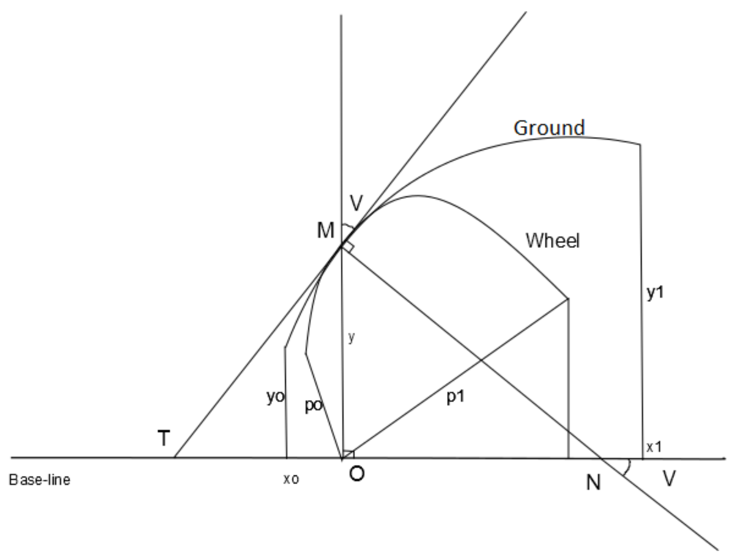

Figure 1: ground and wheel

Example 1 : Rolling circles of Al Tusi-Cardan :
If a circle rolls inside a circle of a circle twice bigger then "any" point on the small circle describes a diameter of the great circle and in this symetrical case no initial condition is necessary.
It is also the definition of the hypocycloid with 2 cusps which is a segment (the diameter of the great circle) described twice after each complete turn. The ground is the circle $x^2 + y^2 = a^2$ and the wheel is the small circle $\rho = a. \cos\theta$. The base-line is a diameter of the circle-ground

Example 2 : Parabola-spiral of Archimede (Cavalieri or G. de St Vincent about 1645).
The Spiral of Archimede can roll inside a parabola in such a way that the pole of the spiral (the wheel) $\rho = a.\theta$ describes the axis (=the base-line) of the parabola $y^2 = 2ax$ (the ground)

The interval of definition of the wheel and of the ground can be limited to $[\theta o, \theta 1]$, $[xo, x1]$ or for θ or x $\in \mathbb{R}$: $]-\infty, +\infty[$.
The two associated curves $C_1(\rho, \theta)$ and $C(x, y)$ have many related properties

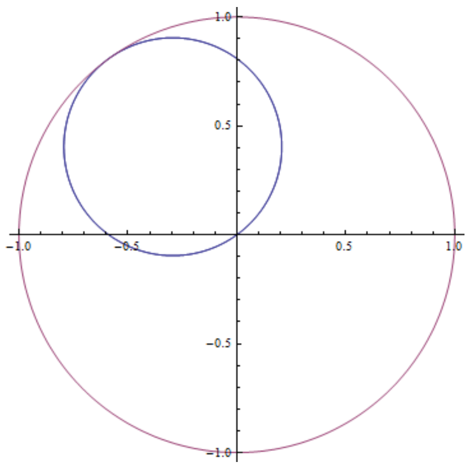

Figure 2: Al Tusi-Cardan (r)-Circle wheel and (2r)-Circle-ground

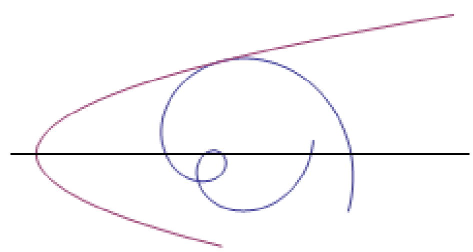

Figure 3: Parabola as the ground - Archimede spiral as the wheel

and will be called Wheel (W) and ground (G) because of the following statement :

Theorem 1 : If the wheel is rolling on the ground the pole of the polar curve (W) describes the xx' axe. With the condition $y_0 = \rho_0$ and the oriented tangent coincides at the beginning of the motion. All the examples respect this condition imposed by the equality of $\tan V$ at corresponding points.
The two arc lengths between two associated points are equal and :

$$ds^2 = \rho^2 d\theta^2 + d\rho^2 = d\rho^2.[1 + \tan^2 V] = \left[\frac{d\rho}{\cos V}\right]^2 = \left[\frac{\rho d\theta}{\sin V}\right]^2$$

.

Theorem 2: (Steiner 1840-Habicht 1882). If the roulette of the pole of a curve (C2) in polar coordinate rolls on a line xx' then the pedal w.r.t. the pole of this curve is associated as a wheel with the roulette and the pole of the pedal describes the axis xx'. The area inside wheel and the one under the ground are

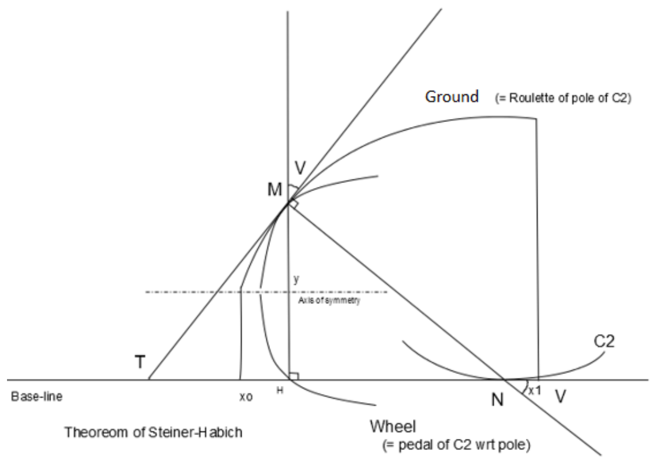

Figure 4: Theorem of Steiner Habich

related by :

$$\frac{1}{2}\int_{\theta o}^{\theta}\rho^2 d\theta = S_w = \frac{1}{2}\times S_r = \frac{1}{2}\int_{xo}^{x} y.dx$$

Theorem 2 is an important property which relates wheels with their pedals and associated grounds:
If a curve in polar coordinates (C_2, M) rolls on a line (D), the pole M describes a roulette. The pedal of (C_2, M) is described by H the projection of O on (D) which is the base-line Δ for the roulette generated by $(C_2), M)$.
The symmetric of this pedal w.r.t. the line parallel to the base, mediatrice of MH in the fixed cartesian plane, is in the position of the wheel. We need two transformations one to pass from the curve (C_2, M) to the wheel [pedal of (C_2, M)] and another (axial symmetry) to place the wheel in the adjusted position tangent to the ground-roulette.

Example 3 : The cardioid is the pedal of a circle w.r.t. a point on the circle, the roulette (on a line) of the same circle for a point on the circle is the cycloid: by theorem of Steiner-Habich the cardioid is a wheel for the cycloid.

We recall some properties of the roulette on a line :

Theorem 3 : When a curve (C_2, M) rolls on a line $x'x$ the angle V of (C_2) is the opposit of the corresponding V of the roulette the curve on which the pole M moves. The two angles have opposit orientation and orthogonal sides.

Angle V is essential in the gregory's transformation and the parametrisation of the plane curves will use V or a simply related angle as the parameter of the curves.

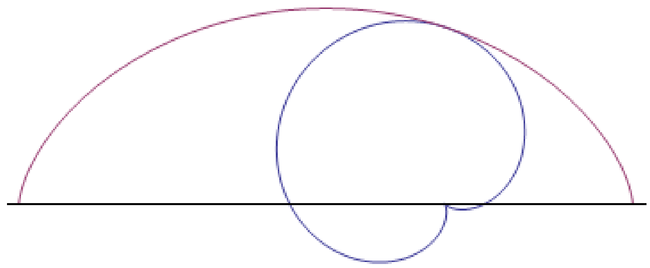

Figure 5: Wheel = cadioid and ground = cycloid

1.1 Duality of plane motion by rolling.

Theorem 4 : When a curve (C_1) in the moving plane rolls on an other curve (C) in the fixed plane the pole of (C_1) describes a curve (C') in the fixed plane linked with (C) : the roulette $(C_1/C) = (C')$. And if we exchange the role of the two planes : (C) rolls on (C_1) this time fixed then the curve (C') linked with (C) passes through the pole of (C_1).

This property of duality is applied most often to the case if the fixed curve is a line.

1.1.1 Definitions of dual elements in the fixed or rolling plane :

1-The wheel (W) is a couple of two objects in a plane : a curve (C_1) and a point O (named the pole) linked to the curve. This couple is defined by the polar equation of the curve (C_1) with respect to O as the pole : $\rho = f(\theta)$
2-The ground (R) is a couple of two objects in a plane : a curve (C) and a line Δ linked to the curve and called the base-line. This couple is defined by the orthonormal cartesian frame $x'x, y'y$ and the equation of the curve (C) in this frame is : $y = f(x)$. The axis x'x is always the base-line.

Gregory's transform (GT) just traducts the fact that two curves for which the equations (1) or (2) - in Cartesian and polar coordinates - are verified have the relation wheel - ground. So if $y = \rho$ and $V_w = V_r$ at the beginning of the motion the two rigid profiles will roll one on the other such that :

1- If the wheel rolls on the ground in the fixed plane then the pole O linked to the wheel runs along the base $(x'x)$,
2- If the ground rolls on the wheel in the fixed plane then the base-line $(x'x)$ linked to the ground pass through the pole of the wheel.
These two facts are dual in the sense defined above.The "base-line" of the ground is the dual element of the pole. A translation in the ground-plane corresponds to a rotation in the wheel-plane.

There is a strong relation between the two curves associated by GT : $y = \rho$,

angle V is the same, the arc length is equal between two positions consequence of the rolling movement if the initial conditions are verified.
The common element of arc length is $ds = \frac{\rho d\theta}{\sin V} = \frac{d\rho}{\cos V}$ and the length between two points :

$$s_r = \int_{x_0}^{x_1} ds_r = \int_{\theta_0}^{\theta_1} ds_w = s_w$$

The corresponding 'Leibniz' triangles MTHN in cartesian and MTON in polar are equal at each instant of the rolling movement of the couple of curves. This triangle is defined by the values y or ρ and the common angle V.

1.1.2 Rotation around O, translation along the Base-line and orthogonal trajectories

The Gregory's transformation makes a connection in the plane between translation along the base-line x'x in the ground-plane and rotation around O in the wheel plane. To find orthogonal curves to wheels $[W_1, O]$ in rotation around O is equivalent to find a ground orthogonal to the corresponding ground translated along the base-line x'x. It suffices to keep the variables ρ or y and fix the new $V = \pi/2 - V$. The rotation around O in wheel plane is the dual of translation in the ground plane along the base-line.

An angle dilation $[\theta \to r.\theta \quad r \in \mathbb{R}]$ of the wheel leads by GT to an affinity along the base-line (x'x axis).
Example : the rhodoneas $[\rho = \cos(r.\theta)]$ are wheels for ellipses w.r.t. x- or y-axis as base-line. Case r=1 is Cardan property. An homothety of the wheel infers by GT an homothety in the ground plane.

1.1.3 The pedal and the Δ-pedal

The positive pedal of a curve (C',O) is the curve generated by the projection H of O on the tangent at the end of ρ. We can define the successive pedals indexed by a parameter p in N.The equation of the pedal is :

$$\rho_p = \rho \sin V \text{ and } \theta_p = \theta + (\pi/2 - V)$$

For the pedal p becomes p+1 and θ becomes $\theta - V + \pi/2$.

Theorem 5 : The angle V of the rolling curve is the same as the corresponding angle of the pedal. This can easily be showed by an argument based on the center of elementary rotation (CER).
The negative pedal (or anti-pedal) is the enveloppe of the perpendicular to the end of ρ. For the anti-pedal p becomes p-1. The equation of the anti pedal is :

$$\rho = \frac{\rho}{\sin V} \text{ and } \theta_{-p} = \theta - (\pi/2 - V)$$

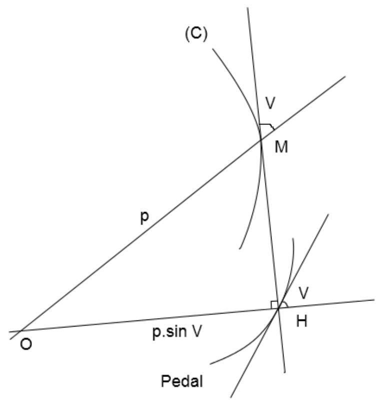

Figure 6: Pedal anti-pedal

Combining the two cases we can define in the same way the successive $\mp p^{th}$ pedal indexed by a parameter p $\in \mathbb{Z}$.

$$\rho_{-p} = \frac{\rho}{\sin V} \text{ and } \theta_{-p} = \theta + p.(\pi/2 - V)$$

We define then another type of pedal :
Def : the Δ-pedal w.r.t. a base-line is the curve described in the plane of a ground (x,y) by the projection of O (foot of the y coordinate) on the tangent at current point M (x,y).
The equation of this Δ-pedal is :

$$x_{\Delta_p} = x - y \sin V \cos V \text{ and } y_{\Delta_p} = y \sin^2 V$$

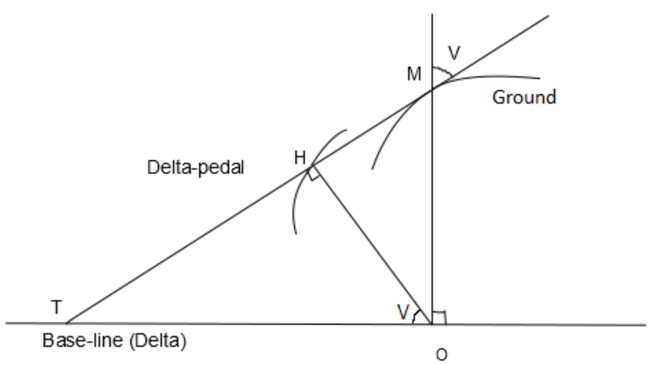

Figure 7: Delta pedal

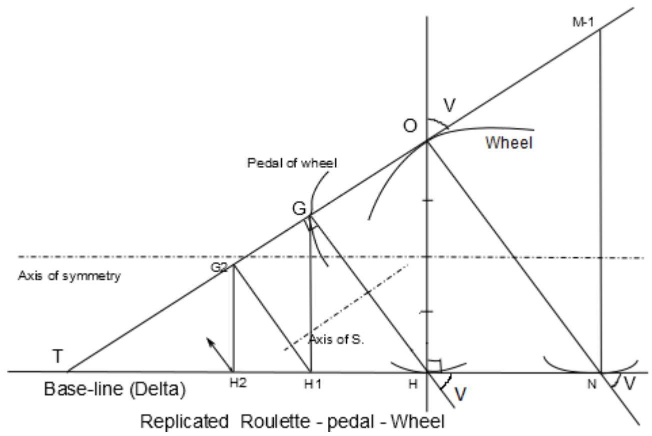

Figure 8: Replicated sequence roulette-pedal-wheel

1.1.4 Replicated sequence Roulette ↦ Pedal ↦ wheel

In the triangle ONHT we have curve (C') rolling on line TN, the roulette is traced by O (ground). Take the pedal of (C') : it is a curve passing at H. Take the symetric of this pedal w.r.t. the mediatrix of OH and the wheel is in the rolling position with common tangent at O. This procedure can be repeated with new wheel wich rolls on the line OT. We use the new triangle HOGT and the rolling curve is the last wheel. Take the pedal, then the symmetric w.r.t. new axis (mediatrix of GH). The same process may continue endlessly. We have created a chain of curves presenting a common angle V and corresponding to the successive pedals/antipedals of the initial curve (C').

2 Some Theorems related to couples (W, G) :

We give here some properties associated to couples (W/G) or conjugated profiles.

Theorem 6 : The wheels $[W_1, O]$, $[W_2, O']$ associated with two parallel base-lines at the distance d, for the same ground (C) are rolling curves about two fixed poles at the same distance d.

Theorem 7: If we fix the pole (i.e. O) of one of the curves of theorem 6 the pole (O') of the other describes a circle of radius = d around O.
Remark : the properties are all depending on an initial position compatible with $\rho_1 + \rho_2 = d$ and common angle V with the rolling (as indicated above for the couple ground/wheel)

Theorem 8 : If the two preceding wheels $[W_1, O]$, $[W_2, O']$ roll simultaneously on a line (it is true for any curve not only a line) then the roulettes of

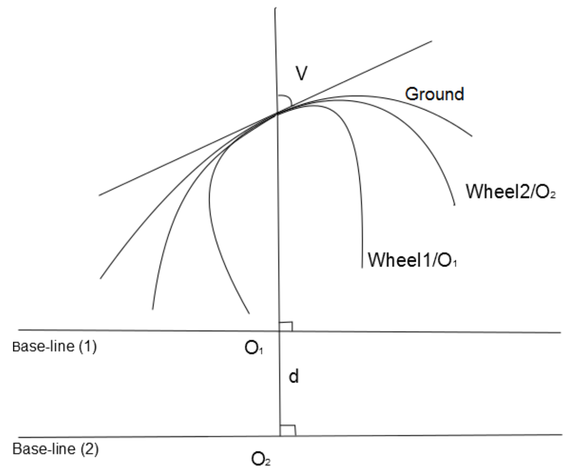

Figure 9: rolling of two wheels and common ground and two base-lines.

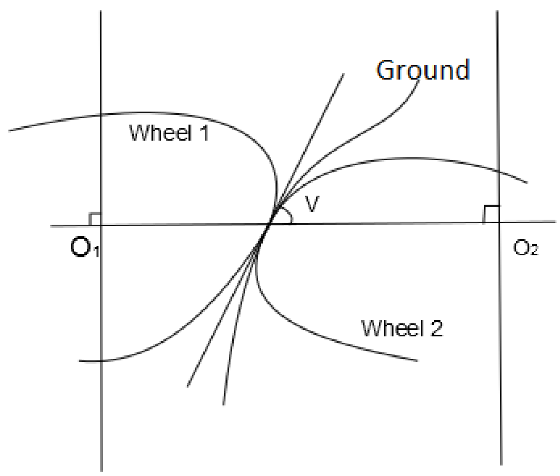

Figure 10: two wheels turning about two poles.

the poles O and O' are parallel curves, at distance d, in the fixed plane and if the ground rolls synchroneously on the same line (true for any curve) then the envelope of the two corresponding base-lines are the two preceding roulettes of O an O'.

Theorem 9 : If the wheel $[W_1, O]$ rolls on the other side of the common tangent, so if O the is symmetric of the point H w.r.t. the current tangent then O describes a well known curve called the anticaustic of the ground for parallel rays of light coming from ∞ in the y direction ($\perp$ to base-line).
The equation of this anticaustic is :

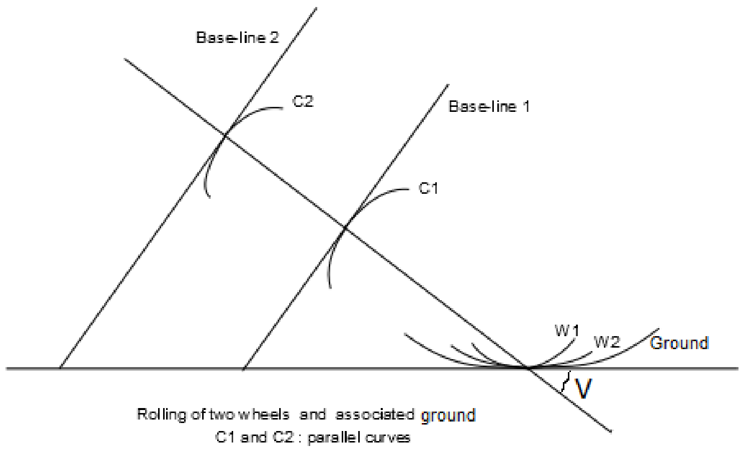

Figure 11: Parallel curves generated by rolling of two wheels and common ground.

$$x_H = x - 2.y \sin V. \cos V \text{ and } y_H = 2.y. \sin^2 V$$

The caustic is simply the evolute of this anticaustic.

3 "Radials", "Manheim curves", wheels and grounds

The equation of Cesaro of a curve (C) is $f(s, R_c) = 0$ where s represents the curvilinear abcisse and R the radius of curvature. Mannheim has proposed a correspondance of curves by using this equation in an orthonormal frame with $x = s$ and $y = R_c$ so $f(x, y) = f(s, R_c)$. The Cesaro equation $f(s, R_c)$ interpreted in the new orthonormal frame $f(x, y)$ with $x = s$ and $y = R_c$ is called the Mannheim curve of (C). The radial of a curve in a polar coordinate system is $\rho = R_c$ and θ angle of position parallel to the radius of curvature.
We have the equivalence :
Theorem 10 : The Mannheim curve (x,y) - with base-line x'x - and the radial (ρ, θ) of all plane curves form a couple (ground / wheel).

4 Curvature and rotation :

For curves in polar parametric coordinates in the plane there is a formula to determinate the radius of curvature R_c depending on ds, dθ and dV :

$$R_c = \frac{ds}{d\theta + dV}$$

This expression shows that there is a perfect symmetry (curvature equivalent to rotation) between the variation of the angle V of the tangent and the variation of rotation angle θ. A deformation of the ground is tranfered to the wheel by Gregory's transformation. If we use $V = u$ as the parameter of the curve $ds = \frac{\rho d\theta}{\sin(V)}$ so the expression of R_c is :

$$R_c = \frac{ds}{d\theta + dV} = \frac{\rho}{[1 + \frac{dV}{d\theta}].\sin(V)}$$

The following formula connects the curvature radius (R_{cR}) of the ground and the one (R_{cW}) of the wheel at the touching point :

$$\frac{R_{cR}}{R_{cW}} = 1 + \left|\frac{d\theta}{dV}\right|$$

In the plane a curve can be define by intrinsic equation of Cesaro $R_c = f(s)$.

5 Singularities of the wheel on y=0 in reverse Gregory's transformation :

The direct Gregory's transformation can be applied to all smooth polar curves (ρ,θ) in the plane to calculate the (x, y) equation of the associated ground. The two equations have no singularity for smooth curves there is one ground up to a translation along the base-line. For the reverse Gregory's transformation the integral may present singularities on the line y=0 (base-line). Given a ground which crosses or is tangent to this singularity (or critical) line we have to study what happens for the corresponding wheel.

The reverse TG (from $G \to W$) gives a wheel generally not closed and with exponential singularities if the ground crosses or is tangent to the critical-line (y=0) at an angle $\neq \pi/2$. This can be shown by approximating the ground by the line tangent at the crossing point. When the ground is a line $y = \frac{1}{\tan V}.x$ then the ground is a logarithmic spiral $\rho = \exp(\theta \tan V)$
So the singularity of the corresponding wheel is exponential and looks like the asymptotic point of the logarithmic spiral. The angle of rotation goes to $\pm\infty$. Actually there is an angular discontinuity at the point of intersection of the ground and the base-line. For line-ground the wheel is composed of two distinct logarithmic spirals one for positive part of the ground and one for negative part. At the crossing of the base-line y=0 the angle diverge to $\pm\infty$.

We can distinguish 5 cases :
1- Crossing with angle α ($\neq 0$ and $\neq \pi/2$) :
then the wheel has an asymptotic point in $\rho = 0$ with a finite length.
2- Crossing with angle $\alpha = \pi/2$:
then the wheel passes throught the pole without singularity.
3- Touching : angle $\alpha = 0$ (tangent) :

then the wheel has an asymptotic point in $\rho = 0$ with a finite length.
4- Touching : angle $\alpha = 0$ (asymptotic/point at ∞ on the base-line) :
then the wheel has an asymptotic point in $\rho = 0$ with an infinite length.
5- The ground does'nt cross the base-line $y = 0$ nothing special here.

5.1 Line-grounds and spiral logarithmic wheels and stairs.

Two particular cases are to be noted for the exponential wheels linked to line grounds : for $V = 0$ the wheel is a line passing through pole O, which corresponds to a line ground orthogonal to the base la line and for $V = \pi/2$ the wheel is the traditionnal wheel $\rho = R$. So we can use the preceding facts to eliminate these singularities of the wheel on line y= 0 by using a simple trick. Just create a stair over the singularity. Its a segment of line which goes through the pole of the wheel and evitates the infinite rotation by crossing the base-line with an angle $\neq \pi/2$. For the ground it looks just like a stair for the crossing.
The line-wheel is associated to the catenary-ground and there are two special cases : the catenary is a horizontal line and the catenary is a vertical double line touching x'x. We find in an other way the preceding stair.

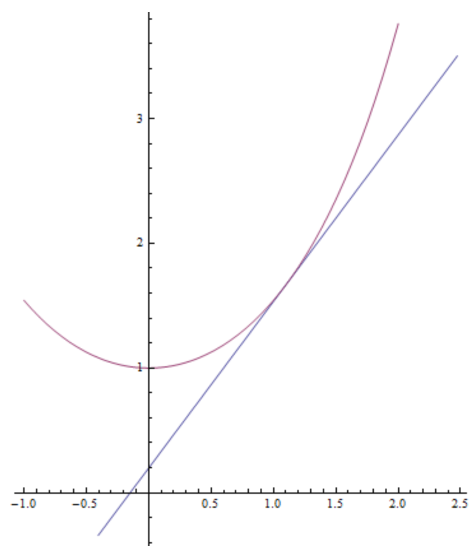

Figure 12: Couple Line-catenary

The simple definition of the Gregory's tranformation and its geometric properties permit to create couples W/G at will by sticking pieces of different curves in any way. Angular points, stairs, standards singularities like cups, double points etc. are supported without difficulty and it is possible to approximate all wheels or grounds by stairs.

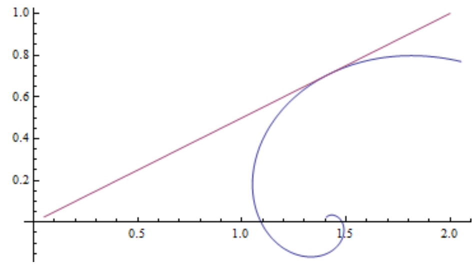

Figure 13: Couple logarithmic Spiral-Line

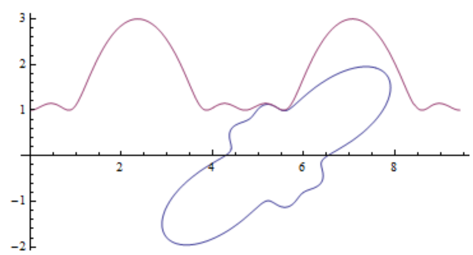

Figure 14: ground-wheel

6 Definition of the generalised angle φ

Gregory's transformation and couple wheel-ground are a generalisation of the Cardan-Al Tusi property. It is possible to define by analogy an angle angle for any plane curve $y = f(x)$ as for the ground $y = f(x)$ by the formula: $Angle = \int_{xo}^{x} \frac{dx}{y}$. Al Tusi property is a special case of the angle at the center of the great circle ($x = \cos t, y = \sin t$). This angle is equal to the opposit of the corresponding angle of the small circle for two different position because of the relation :

$$-Angle = \int_{\theta o}^{\theta} d\theta = \theta - \theta_o$$

$$\int_{xo}^{x} \frac{dx}{y} = \int_{t0}^{t} \frac{-\sin t.dt}{\sin t} = \int_{t0}^{t} -dt = t_o - t = Angle$$

So the general angle for the wheel (ρ, θ) is the opposite of the one mesured on the ground $y = f(x)$ for two corresponding points by rolling.

There is a condition for y because the line y=0 is a singularity line for the wheel (polar angle becomes infinite) except when the ground crosses the line y=0 orthogonally and it is the case in the Cardan-Al Tusi property.

This angle is similar to those introduced in the theory of elliptic functions $Angle = \int_{xo}^{x} \frac{dx}{y}$. Here the analogy is readable and the GT^{-1} is a geometric interpretation of these angles.

The elliptic function $\int_0^{\varphi} \frac{dx}{y} \longrightarrow y = \sqrt{1-x^4} \rightarrow y^2 + x^4 = 1$ and we have :

$$Angle = \int_0^{\varphi} \frac{dt}{\sqrt{1-t^4}} = Arcsl(\varphi)$$

7 grounds, wheels and lengths of algebraic curves.

Leonard Euler in his work (1781) on curves which have same arc-length as the circle had found a class of curves which can be linked with the Gregory's transform. He found curves equivalent to curves-wheels for a circle ground not intersecting the base-line (see Part II).
J.A. Serret has examined the problem of L. Euler for the circle with algebraic means in a paper of 1852 in the Journal de l'Ecole Polytechnique. He obtained a general algebraic solutions (but it is not sure all solutions because the problem is widely undefinite) and some of those mentioned in this work can be studied with the geometric tool of the reverse Gregory's transformation.

In n-dimensions space, as proved by Liouville, there is only a group of conformal transformations (inversion group similar to Moebius group in the plane). But in dimension 2 we have Moebius group and the conformal mapping by function of one complex variable : this is a specificity of dimension 2.

This article is the 1st part on a total of 6 papers on Gregory's transformation and related topics.

Part I : Gegory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalisation of sinusoidal spiral and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and grounds
Part VI : Catalan's curve.
There are also two papers I have published in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (Nr 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (Nr 364 juin1988).There is an in english translation.

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