ANALLAGMATIC SPIRALS
PURSUIT CURVES
HYPERBOLIC-TANGENTOID SPIRALS
β-CURVES
Part - VII

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21/10/2014

Abstract

By analogy with sinusoidal spirals and Ribaucour curves defined with trigonometric functions we use Mc Laurin and pedal transformations to study classes of plane curves : tangentoid and anallagmatic spirals defined with hyperbolic functions and gudermanian. We examine the grounds associated to these curves as wheels by the Gregory's transformation and orthogonal trajectories and describe some special families of curves depending of two integer indexes.

1 Plane curves with an arc length ds integrable by hyperbolic functions

Among plane curves we find classes, like sinusoidal spirals $\rho = \sin^n(\frac{\theta}{n})$, for which the expression the element of arc can be expressed by elementary functions.
We use an analogy between tangentoid curves in polar coordinates $\rho^n = \tanh(\frac{\theta}{2n})$ and $\rho^n = \coth(\frac{\theta}{2n})$ to find families of curves generalized tangentoid curves in a similar method as for sinusoidal spirals in part III.
The angle V between $\overrightarrow{OM}$ and oriented tangent at M is given by the formula: $\tan V = \frac{\rho . d\theta}{d\rho} = \sinh\theta$ so V is the gudermanian of θ and it is a difference with the generalized sinusoidal spirals case. The Gudermanian function, which links real circular and real hyperbolic functions, gives an equivalence between trigonometric functions and hyperbolic functions of parameter u, the hyperbolic argument.
Formulas like :

$$\cosh^2 u = 1 + \sinh^2 u \quad \text{or} \quad \frac{1}{\cosh^2 u} + \tanh^2 u = 1$$

equivalent to :

$$1/\cos^2 V = 1 + \tan^2 V \quad \text{or} \quad \cos^2 V + \sin^2 V = 1$$

and

$$\tan V = \sinh(p.u)$$

give the means to find curves which have simple expressions for the arc length of curves parametrized by hyperbolic functions.

1.1 Some properties of gudermanian functions

The gudermanian is defined by the following equivalent equations :

$$\sinh u = \tan V$$

or

$$\cosh u = 1/\cos V$$

or

$$\tanh u = \sin V$$

or

$$\tanh(u/2) = \tan(V/2)$$

The direct gudermanian is :

$$V = Gd(u) = \arctan(\sinh u)$$

and the inverse gudermanian :

$$u = Gd^{-1}(V) = \arg\sinh(\tan V)$$

equivalent to the above formulas. The function Gd(u) is defined in $]-\infty, +\infty[$ and is in bijection with $]-\pi/2, +\pi/2[$. Gd^{-1} is defined on $]-\pi/2, +\pi/2[$ with the same precautions as for inverse trigonometric functions.
Other formulas are :

$$Gd(u) = \int_0^u \frac{dt}{\cosh t} \quad \text{and} \quad Gd^{-1}(V) = \int_0^V \frac{dt}{\cos t}$$

1.2 The tangentoids spirals : $\rho = \tanh(\theta/2)$ and $\rho = \coth(\theta/2)$

These two curves that I call tangentoid and cotangentoid are transformed one in the other by inversion w.r.t. the pole O. We have :

$$\tan(V) = \frac{\rho.d\theta}{d\rho} = \frac{2.\tanh(\theta/2)}{1 - \tanh^2(\theta/2)} = \sinh(\theta)$$

$$\tan(V') = \frac{\rho.d\theta}{d\rho} = \frac{2.\coth(\theta/2)}{1-\coth^2(\theta/2)} = -\sinh(\theta)$$

The arc length is given by :

$$s(0,\theta) = \theta - \tanh(\theta/2)$$

or

$$s(a,\theta) = [\theta - \coth(\theta/2)]_a^\theta \qquad a.\theta > 0$$

So these arc lengths are given by elementary functions of θ. The angle $V = Gd(\theta)$ since $\tan V = \sinh \theta$.

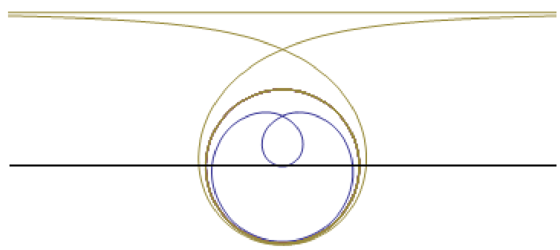

Figure 1: Tangentoid and Cotangentoid with common asymptotic circle $\rho = 1$

1.3 Two transformations : Mc Laurin of order n and pedal/anti-pedal of order p

1.3.1 Mc Laurin's transformation :

Mc Laurin's transformation applied to curves in polar coordinates $(\rho\,,\theta)$ gives equations of transformed curves from the parametric polar equations of an initial curve. We suppose that $n \in \mathbb{Z}$, and set :

$$\rho_{McL} = \rho^n \quad \text{and} \quad \theta_{McL} = n.\theta$$

This is equivalent in transformation $Z = f(z) = z^n$ of the complex plane and it is a conformal transformation. So the angle V is preserved for the new curve. If $z = \rho \exp(\imath\theta)$ in the complex plane then $z^n = \rho^n \exp(\imath n\theta)$. We identify the complex plane $\mathbb{C}$ and the euclidean plane then the module $\rho_{McL} = \rho^n$ and $\theta_{McL} = n.\theta$.
For n=-1 Mc Laurin's transformation is the inversion. In the complex plane this $z \to 1/z$ is an inversion associated with an axial symmetry. Beginning with the tangentoid $\rho = \tanh(\theta/2)$ or $\rho = \coth(\theta/2)$ we get this way the new class similar to sinusoidal spirals but with hyperbolic functions :

$$\rho^n = \tanh\frac{n.\theta}{2} \qquad \text{or} \qquad \rho^n = \coth\frac{n.\theta}{2}$$

1.3.2 Pedal transformation :

The pedal of a given a curve in polar coordinates is the curve described by the projection of pole O on the tangent at the current point of the first curve. The angle V is the same for two corresponding points of the curve and its pedal. The parametric polar equations of the pedal of a curve (ρ_0, θ_0) given in polar coordinates, are:

$$\rho_1 = \rho_0 \sin V \text{ and } \theta_1 = \theta_0 - (\pi/2 - V)$$

The equation of order p pedal (p^{th}) is :

$$\rho_p = \rho_0 \sin^p V \text{ and } \theta_p = \theta_0 - p.(\pi/2 - V)$$

These curves are often defined up to a rotation around the pole O the angle $\pi/2$ may be forgotten. The anti pedal ($pedal^{-1}$) is defined in the same way so successives pedals or anti-pedals are defined in $\mathbb{Z}$. Multiply by $\sin V$ and substract V to the angle θ for the pedal, divide ρ by $\sin V$ and add V to the angle θ for the anti-pedal.
The circle $\rho = 1$ is self transformed by the Mc Laurin and the pedal transformations with the pole at the center of the circle.
Applied to tangentoids we must replace the functions $\sin V$ by $\tanh \theta$ and $\cos V$ by $1/\cosh \theta$ using the gudermanian function $\tan V = \sinh \theta$ or $V = Gd(\theta)$.

$$\rho = \tanh \frac{\theta}{2}. \tanh^p \theta \qquad \text{and} \qquad \phi = \theta + p.Gd(\theta) - p.\pi/2$$

In these formulas the angle θ is the parameter used to represent the polar parametric equations $[\rho, \phi]$. In the following u is the parameter and we keep θ for the polar angle.

1.4 Products of Mc Laurin and pedal transformations w.r.t. the same pole

The angle V is common to all the curves so it is possible to include the two transformations in a single formula by taking the p^{th} pedal of a n^{th} tangentoid spiral.
If we apply these two transformations to the above tangentoid spiral

$$\rho = \tanh(\theta/2) \text{ or } \rho = \coth(\theta/2)$$

we obtain the class of the generalized tangentoid spirals by analogy with generalized sinusoidal spirals reviewed in part III :

$$\rho = \tanh^n \frac{u}{2}. \tanh^p u \qquad \theta = n.u + p.Gd(u) \qquad \text{and } \tan V = \sinh u$$

2 Curves with angle : $\theta = n.u + p.Gd(u)$

We search for curves which verify $\tan V = \sinh(k.u)$ or $\tan V = 1/\sinh(k.u)$ - k* will be for the latter - so that the element of arc length is given by elementary functions. We use the same method as in part III and suppose that the angle θ in parametric polar equations is in the form : $\theta = n.u + p.Gd(u)$ and we calculate function ρ with condition $\tan V$ is equal to $\sinh(k.u)$. We get solutions T for tangentoids in polar parametric coordinates by integration of the following equation :

For k=1 :

$$\tan V = \frac{\rho.d\theta}{d\rho} = \sinh u$$

$$d\theta = [n + p./\cosh u]du$$

$$\sinh(u) = \frac{\rho[n + p/\cosh u]du}{d\rho}$$

$$\int \frac{d\rho}{\rho} = \int \frac{(n + p/\cosh u)}{\sinh u}du$$

$$\int \frac{d\rho}{\rho} = \int \Big[\frac{n}{\sinh u} + \frac{p}{\sinh u.\cosh u}\Big]du$$

$$\ln \rho/C = n.\ln(\tanh u/2) + p.\ln(\tanh u)$$

We get the general parametric polar equations (C=1) :

$$\rho = \tanh^n(u/2).\tanh^p u \qquad \theta = n.u + p.Gd(u) \qquad \text{and } \tan V = \sinh u$$

The formulas depend on two parameters (n=Mc Laurin index, p=pedal index) and include the family of sinusoidal spirals if (p=p, n=0) - with the help of Guderman function -. The general formulas are :

$$T_k(n,p) \quad \text{for} \quad \tan V = \sinh(k.u) \quad \text{and} \quad \theta(u) = n.u + p.Gd(u)$$

$$\rho = e^{\int \frac{[n+p/\cosh u]}{[\sinh(k.u)]}du}$$

$$T_{k*}(n,p) \text{ for } \quad \tan V = 1/\sinh(ku) \quad \text{and} \quad \theta = n.u + p.Gd(u)$$

$$\rho = e^{\int [n+p/\cosh u].[\sinh(k.u)]du}$$

The results for the first values of the integer parameter k or k* from -1 to 3 are listed below.

$k = 1 \to T_1(n,p)$:

$$\rho = \tanh^n(u/2).\tanh^p u$$

These are the curves of section 1.4 above.

$k* = 1 \to T_{1*}(n, p)$:

$$\rho(u) = e^{n\cosh u}.\cosh^p u$$

$k = 2 \to T_2(n, p)$:

$$\rho(u) = e^{\frac{p}{2.\cosh u}} \tanh^{\frac{p}{2}}(u/2).\tanh^{\frac{n}{2}}(u)$$

$k* = 2 \to T_{2*}(n, p)$:

$$\rho(u) = e^{n.\cosh^2 u + 2p.\cosh u}$$

$k = 3 \to T_3(n, p)$:

$$\rho(u) = [\cosh(u/2).\sinh(u/2)]^{p/3}.\tanh^{n/3}(u/2).$$
$$.(1{-}4.\cosh^2 u)^{-2.p/3}.\left[\frac{1 + 2.\cosh u}{1 - 2.\cosh u}\right]^{n/3}.\cosh^p u$$

$k* = 3 \to T_{3*}(n, p)$:

$$\rho(u) = \cosh^{-p} u.e^{\frac{n}{3}\cosh 3.u + p.\cosh 2.u}$$

These formulas give a two parameters family of curve linked by Mc Laurin (n) and pedal (p) transformations such that $\tan V = \sinh(k.u)$ (k) or $\tan V = 1/\sinh(k.u)$ (k*).

3 Curves with angle θ= $n.\cosh u + p.Gd(u)$

In this section we use the same requisition for the angle V of the curves and we impose $\tan V = \sinh(k.u)$ (or $\tan V = 1/\sinh(k.u) \to k*$) but we replace u by $\cosh u$ for the first term of the angle expression. We use parameter angle u as the variable and by the same computation as in the previous section to get solutions S for spirals in polar parametric coordinates by integration of the following equation :

$$\tan V = \frac{\rho.d\theta}{d\rho} = \sinh(ku) \qquad (k)$$

Or

$$\tan V = \frac{\rho.d\theta}{d\rho} = \frac{1}{\sinh(ku)} \qquad (k*)$$

The integration of the preceding equations (for integers k, n, p) needs only elementary transcendental functions.

The general formulas are :

$S_k(n,p)$ for $\tan V = \sinh(k.u)$ and $\theta u = n\cosh u + p.Gd(u)$

$$\rho = e^{\int \frac{[n.(\sinh u)+p/\cosh u]}{\sinh(k.u)} du}$$

$S_{k*}(n,p)$for $\tan V = 1/\sinh(ku)$ and $\theta = n\cosh u + p.Gd(u)$

$$\rho = e^{\int [n.(\sinh u)+p/\cosh u].\sinh(k.u)du}$$

The results for the first values of the integer parameter k or k* from 1 to 3 are listed below.

$k = 1 \rightarrow S_1(n,p)$:

$$\rho(u) = e^{n.u}.\tanh^p u$$

$k* = 1 \rightarrow S_{1*}(n,p)$:

$$\rho(u) = e^{\frac{n}{4}(\sinh 2.u - 2u)}.\cosh^p u$$

$k = 2 :\rightarrow S_2(n,p)$

$$\rho(u) = e^{(n/2).Gd(u)+\frac{p}{\cosh u}}.\tanh^{p/2} u/2$$

$k* = 2 :\rightarrow S_{2*}(n,p)$

$$\rho(u) = e^{(2n/3).\sinh^3 u + 2.p.\cosh u}$$

$k = 3 :\rightarrow S_3(n,p)$:

$$\rho(u) = \sinh^{p/3} u.\cosh^p u.[1+2.\cosh(2u)]^{-2p/3}.\left[e^{\frac{n}{\sqrt{3}}.\arctan[\frac{\tanh u}{\sqrt{3}}]}\right]$$

$k* = 3 :\rightarrow S_{3*}(n,p)$:

$$\rho(u) = \cosh^{-p} u.e^{p.\cosh(2u)-(n/4)\sinh(2u)+(n/8)\sinh(4u)}$$

4 Some subclasses of $T_k(n,p)$ and $S_k(n,p)$

The expressions for ρ needs only elementary functions. The two classes of general curves $T_k(n,p)$ and $S_k(n,p)$ have an element of arc length expressed, using the corresponding equalities of gudermanian function, by :

$$ds = d\rho/\cos V = \rho d\theta/\sin V \quad \text{with} \quad V = \arctan[\sinh(ku)] = Gd(ku)$$

$$\boxed{ds = d\rho.\cosh(ku) = \rho d\theta/\tanh(ku)}$$

The sinusoidal spirals are a special case of the $T_k(n,p)$ and $S_k(n,p)$ since n=0 and p=p gives $\rho = \cos^p(\theta/p)$ and $V = \theta/p$.
Two cases in the classes are fairly simple :
$T_1(n,p)$:

$$\theta = n.u + p.Gd(u)$$
$$\rho = \tanh^n(u/2).\tanh^p u$$

and $S_1(n,p)$:

$$\theta = n.\cosh u + p.Gd(u)$$
$$\rho(u) = e^{n.u}.\tanh^p u$$

Some particular classes of curves extracted from preceedings equations are of peculiar interest :
- Tangentoid spirals.
- Anallagmatic spirals

The first subclass leads by the Gregory's transformation to curves I call β-curves as the ground and the last one to pursuit curves as the ground.

4.1 Anallagmatic spirals

Among the curves $S_1(n,p)$ with $\tan V = \sinh u$ we find sinusoidal spirals when n=0 and when p=p another class called anallagmatic spirals, their equations are :

$$\rho(u) = e^{n.u} \text{ and } \theta = n.\cosh u \qquad \tan V = \sinh u$$

These curves have interesting properties :

4.2 Invariance in a pole inversion of power 1

In equations of anallagmatic spirals we can eliminate u between $\rho(u)$ and $\theta(u)$ and we get :

$$\rho^{2/n} - (2\theta/n).\rho^{1/n} + 1 = 0$$

So $\rho_1.\rho_2 = 1$: the spirals are invariant in the inversion of center at the pole O and power 1. For n=1 the deferent is an involute of a circle, the anti-pedal of the spiral of Archimede.

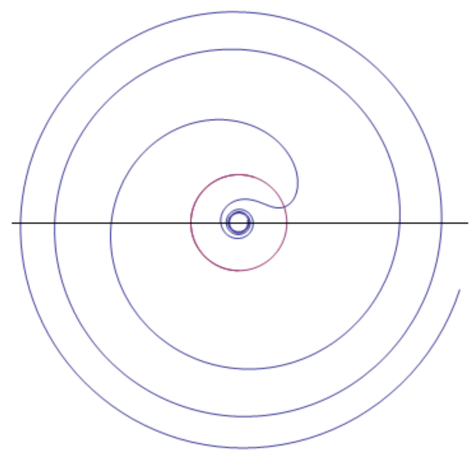

Figure 2: An allagmatic spiral (n=1/3) and circle $\rho = 1$

4.3 Wheels for the Bouguer curves of pursuit as the ground

Using the equations of anallagmatic spiral $\rho(u) = e^{u/k}$ and $\theta = k.\cosh u$ (we set k=1/n in the computations) and we can show that they are related to the pursuit curves when the prosecutor runs after a prey moving along a line with constant speeds. These curves were studied by Bouguer in the eighteenth century and have a simple arc length when the ratio between the speeds is a rational number. These spirals are wheels corresponding to pursuit curves as the grounds and asymptote as the base line. With Gregory's direct tranformation :

$$y = \rho = e^{u/k} \qquad x = \rho.d\theta = \int e^{u/k}.\frac{1}{k}.\sinh u.du = \int \frac{1}{k}.e^{u/k}.\frac{[e^u - e^{-u}]}{2}.du$$

so the parametric equations of the ground are :

$$x = \frac{1}{2}.\Big[\frac{1}{k+1}.e^{u.(k+1)/k} + \frac{1}{k-1}e^{-u.(k-1)/k}\Big] \text{ and } y = e^{u/k}$$

$$x = \frac{1}{2}.\Big[\frac{1}{k+1}.y^{k+1} + \frac{1}{k-1}y^{1-k}\Big]$$

This is the usual equation of pursuit curves with x'x as the line along which the prey moves. If we set $p = \frac{-k}{k+1}$ and $y^{k+1} = t$ we get the new parametric equations :

$$x = t - \frac{t^{2p+1}}{2p+1}, \quad y = \frac{2}{p+1}.t^{p+1}$$

$$dx = [1 - t^{2p}].dt \text{ and } dy = 2.t^p \text{ so } \quad ds^2 = [1 + t^{2p}]^2.dt^2$$

$$s - s_0 = \int_{t_0}^{t} [1 + t^{2p}].dt = \Big[t + \frac{t^{2p+1}}{2p+1}\Big]_{t_0}^{t}$$

This confirms that pursuit curves have a simple algebraic expression for arc length when p (or k) is rational. It is a general property of curves of direction (Salmon-Laguerre) which are caustics by reflection of algebraic curves for parallele light rays. Pursuit curves are caustic by reflecion of $y = x^k$ for light rays parallele to x'x or y'y (Gomez-Teixeira).
Note that pursuit curves defined by a base-line and a geometric condition are symmetric w.r.t. to this line and there are solutions on each side of the axis x'x on which moves the prey. Initial conditions fix the choice between the solutions and place the entire curve on one side.

4.4 Anallagmatic spirals associated with singular pursuit curve and Tschirnhausen's cubic

We list some examples for small values of n:
n= 1 correponds to the singular pursuit curve (equal speed of the prosecutor

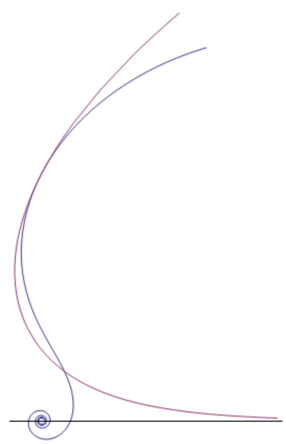

Figure 3: Wheel : $\rho = e^u \quad \theta = \cosh u$ and Singular pursuit curve ground

and of the prey) and parametic equations are :

$$x = (1/4).[e^{2.u} - 2.u] \qquad y = e^u$$

The associated wheel for asymptote as the base line is the anallagmatic spiral with equations :

$$\theta = \cosh u \text{ and } \rho = e^u$$

$$\theta = \frac{1}{2}.\left[\rho + \frac{1}{\rho}\right]$$

Pursuit curve n=2 is the Tschirnhausen's Cubic :

$$x - x_o = t - t^3/3 \text{ and } y = t^2$$

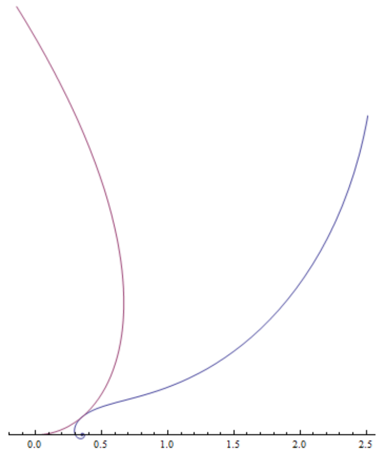

Figure 4: Wheel [$\rho = t^2 \quad \theta = t + 1/t$] and Tschirnhausen's cubic ground

The wheel associated to the x'Ox axis as the base-line is a special case of anallagmatic spirals and is given in parametric polar coordinates by :

$$\theta = t + \frac{1}{t} \text{ and } \rho = t^2$$

or

$$\theta = 2.\cosh u \text{ and } \rho = e^{2.u}$$

Pursuit curve n=3:

$$x - x_o = \frac{3}{8}.\left[t^4 - 2.t^2\right] \qquad y = t^3$$

The corresponding wheel is :

$$\theta = (3/2)[t + \frac{1}{t}] \text{ and } \rho = t^3$$

or

$$\theta = 3.\cosh u \text{ and } \rho = e^{3.u}$$

4.5 Orthogonal trajectories of pursuit cuves translated along x'x axis : $\beta-$curves

The family of pursuit curves parametrized by x_o (translation along x'x) :

$$x - x_o = \frac{1}{2}.\Big[\frac{1}{k+1}.y^{k+1} + \frac{1}{k-1}y^{1-k}\Big]$$

are solutions of the differential equation :

$$dx = \frac{1}{2}.\Big[y^k - y^{-k}\Big].dy$$

The orthogonal trajectories of pursuit curves are solutions of the differential equation derived from the preceeding with substitution $dy/dx \to -dx/dy$

$$dx = \frac{-2.dy}{y^k - y^{-k}}$$

or

$$dx = \frac{-2.y^k.dy}{y^{2k} - 1}$$

I call β-curves the solutions of this last differential equation, the orthogonal trajectories of the pursuit curves translated parallely to x'x. We list the first cases :

k= 1 : orthogonal trajectories of singular pusuit curve translated along x'x :

$$x - x_o = \ln[1 - y^2] \text{ or } x - x_o = \ln[y^2 - 1]$$

k=1/2 : orthogonal trajectories of Tschirnhausen's cubics $y = t^2$, $x = t - t^3/3$ translated along x'x :

$$x - x_o = 4[\tanh\alpha - \alpha] \quad y = \tanh^2\alpha \text{ or } x - x_o = 4[\alpha + \coth\alpha] \quad y = \coth^2\alpha$$

k=1/3 : orthogonal trajectories of pursuit curve of ratio=1/3 translated along x'x:

$$x - x_o = 6[\ln(\cosh\alpha) - \frac{1}{2}.\tanh^2\alpha] \quad y = \tanh^3\alpha$$

or

$$x - x_o = 6[\ln(\sinh\alpha) - \frac{1}{2}.\coth^2\alpha] \quad y = \coth^3\alpha$$

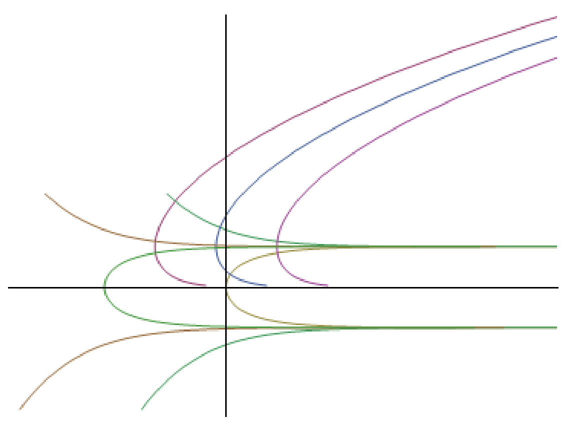

Figure 5: Pursuit curves and β-curves are orthogonal trajectories for translation along x'x

4.6 Orthogonal trajectories of anallagmatic spirals rotating around the pole.

We search the orthogonal trajectories of the tangentoids spirals depending on one parameter θ_0 the angle of rotation around the pole.

The classic method to find the orthogonal trajectories of the curves :

$$\rho^n = \tanh[n.(\theta - \theta_o)/2] \qquad \text{or} \qquad \rho^n = \coth[n.(\theta - \theta_o)/2]$$

give for the first one :

$$n.\rho^{n-1}.\rho'_\theta = (n/2).[1 - \tanh^2 n.(\theta - \theta_0)/2]$$
$$2.\rho^{n-1}.\rho'_\theta = 1 - \rho^{2n}$$
$$d\theta = \frac{2.\rho^{n-1}.d\rho}{1 - \rho^{2n}}$$

With the substitution : $\rho'_\theta \mapsto -\rho^2/\rho'_\theta$ the differential equation becomes :

$$-2.\rho^{n+1} = [1 - \rho^{2n}].\rho'_\theta$$

or

$$d\theta = \frac{1 - \rho^{2n}}{-2.\rho^{n+1}}.d\rho$$
$$d\theta = \frac{\rho^{n-1} - \rho^{-n-1}}{2}.d\rho$$

The solutions are the anallagmatic spirals, set $\rho = e^{u/n}$ gives :

$$\theta - \theta_o = \frac{1}{n}.\cosh u$$

So we have the property illustrated on fig. 6 and 7. And we get this way two orthogonal families of couples ground/wheel :
- Pursuit curves (ground) - Anallagmatic spirals (wheel) and
- β-curves (ground) - Tangentoid spirals(wheel) which are orthogonal translated along x'x for the grounds and rotated around pole for the wheels for a given n. And the element of arc length of all these curves if n is an integer have a rational expression since $\tan V = \sinh u$ with u as the parameter. Some grounds associated with the tangentoid spirals as wheels are the curves listed at the end of the preceeding subsection.

5 Generalized pursuit curves and generalized β-curves.

In part III we defined generalized Ribaucour curves as grounds associated with the generalized sinusoidal spirals.
In the same way we can define the generalized pursuit curves as grounds for the

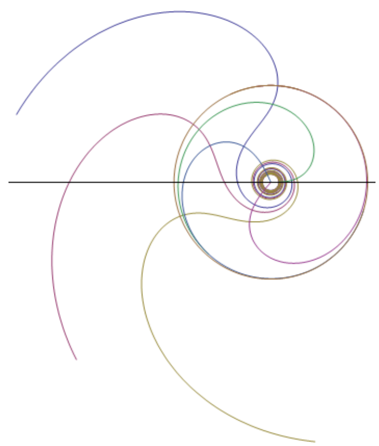

Figure 6: Tangentoid and anallagmatic spirals are orthogonal trajectories for rotation around the pole

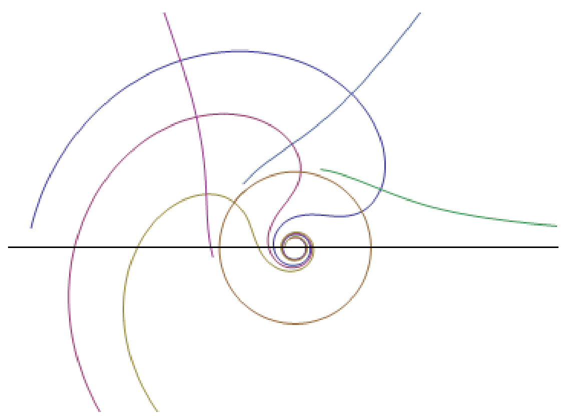

Figure 7: Cotangentoids and anallagmatic spirals are Orthognal trajectories for rotation around the pole

generalized anallagmatic spirals as wheels :

$T_k(n,p):\quad \theta = n.u{+}p.Gd(u)\quad$ and condition : $\quad \tan V = \sinh(k.u)\quad$ or $\quad \tan V = 1/\sinh(k.u)$

We have seen above the special case of $\beta-$curves as grounds for tangentoid wheels (n=n, p=0).
Another class of generalized β-curves can be defined as the grounds associated to the generalized tangentoid spirals as wheels :

$S_k(n,p):\quad \theta = n.\cosh u{+}p.Gd(u)\quad$ and condition : $\quad \tan V = \sinh(k.u)\quad$ or $\quad \tan V = 1/\sinh(k.u)$

6 References

This article is the X^{th} part on a total of 10 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, closed or periodic curves and Gregory's transformation.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).
There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr
Books :
- Courbes geometriques remarquables 'H. Brocard , T. Lemoine) Blanchard Paris 1967 (3 tomes)
- Traite des courbes speciales remarquables (F. Gomez Teixeira) Chelsea New York 1971 (3 tomes)
- Nouvelles annales de mathematiques (1842-1927) Archives Gallica
- Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
- Geometriae pars universalis (James Gregory) Padova 1668.