# INTRINSICALLY DEFINED PLANE CURVES, CLOSED OR PERIODIC CURVES AND GREGORY'S TRANSFORMATION - Part X -

C. Masurel

23/04/2014

**Abstract**

Cesaro defined plane curves by its intrinsic equation : a relation between the radius of curvature ($R_c$) and the arc length s counted from a fixed point on the curve. We give some examples. The curvature function $\kappa(s) = \frac{1}{R_c}$ defines a unique curve in the plane. We also recall some conditions, when curvature has period T, for the closure of the resulting curve.

## 1 Intrinsic equation of Cesaro

Cesaro's intrinsic equation links the radius of curvature ($R_c$) and the arc length (or curvilinear abcissa) and defines a unique curve in the plane. It can be proved that a function relating the two parameters defines a unique curve in the plane up to a motion or a dilation (equivalence up to the group of plane similarities). The curvature determines the curve. We consider as "different" curves that cannot be transformed one in the other by a plane similarity. The intrinsic equation can be presented in the implicit form : $F(R_c, s)$ or explicit forms $R_c = f(s)$, $s = f^{-1}(R_c)$ but we shall most often use $R_c = f(s)$.
Many authors use the curvature as a variable $\kappa = \frac{1}{R_c}$ that gives simple expression for the leading angle V measured from a fixed direction :

$$V = \int \kappa(s).ds \qquad \text{since} \qquad \kappa(s) = \frac{1}{R_c} = \frac{dV}{ds}$$

An intrinsic equation is simple method to create curves in the plane but it can lead to very cahotic results as curves covering the entire plane as a consequence of irrationality ([D],[E ]).

### 1.0.1 Mannheim curve

Cesaro's equation of a curve (C) is $f(s, R_c) = 0$ where s represents the curvilinear abcissa (or arc length) and $R_c$ the radius of curvature at s. The Mannheim curve associated to (C) is the curve graph of the intrinsic equation in an $(x, y)$ orthonormal frame. The arc length is measured along the curve from a given origin. So this is the Cesaro equation $f(s, R_c)$ of the initial curve in the orthonormal frame (O, x, y) : $f(x, y)$ with $x = s$ and $y = R_c$.

### 1.0.2 Radial of a plane curve

The radial of a curve is another curve in a polar coordinates frame with $\rho = R_c$ and $\theta$ is the angle of position parallel to the radius of curvature of the initial curve.

### 1.0.3 Couple Radial-Mannheim and couple Wheels and grounds

We have the equivalence :
Theorem (Balitrand 1915) : The Mannheim curve (x,y) - with base-line x'x - and the radial ($\rho$, $\theta$) of a given plane curve form a couple (ground/wheel). When the curve is defined by its intrinsic equation we deduce when it is possible the couple of associated curves (ground and wheel) by the Gregory's transformation $TG$ / $TG^{-1}$: $x = s, \quad y = R_c = \frac{1}{\kappa(s)}$ for the ground and : $\theta = \int \frac{ds}{R_c} = \int \kappa(s).ds, \quad \rho = R_c = \frac{1}{\kappa(s)}$ for the wheel.

## 1.1 Whewell equation

The Whewell equation uses two parameters to define the curve : the arc length s and the angle V which is counted between the y-axis in the plane and the oriented current tangent to the curve at s. It is not really intrinsic since V measured from an arbitrary fixed direction (or y-axis) outside. Whewell equation can be written $V = \int \kappa(s)ds = V(s)$ and we have :

$$\frac{dx}{ds} = \sin V \qquad \frac{dy}{ds} = \cos V$$

$$\frac{dx}{dy} = \frac{\frac{dx}{ds}}{\frac{dy}{ds}} = \tan V$$

$$x - x_o = \int R_c.\sin V.dV \qquad y - y_o = \int R_c.\cos V.dV$$

$$x - x_o = \int \frac{1}{\kappa(s)}\sin V.dV \qquad y - y_o = \int \frac{1}{\kappa(s)}\cos V.dV$$

$$x - x_o = \int \sin[V(s)].ds \qquad y - y_o = \int \cos[V(s)].ds$$

Intuitively this means that we can affect to each point at position s a direction V as if we were the pilot on the curve choosing the trajectory : we need the curvature $\kappa(s)$. If we use the Gregory's point of view you look from the centre of curvature and examine the curve at a distance $R_c$, how it must be bent and how much we must straighten it to transform it in the x-line (x-axis) with a constant infinite radius of curvature. The element angle of rotation is $dV = \frac{ds}{R_c}$. Both points of view are complementary.

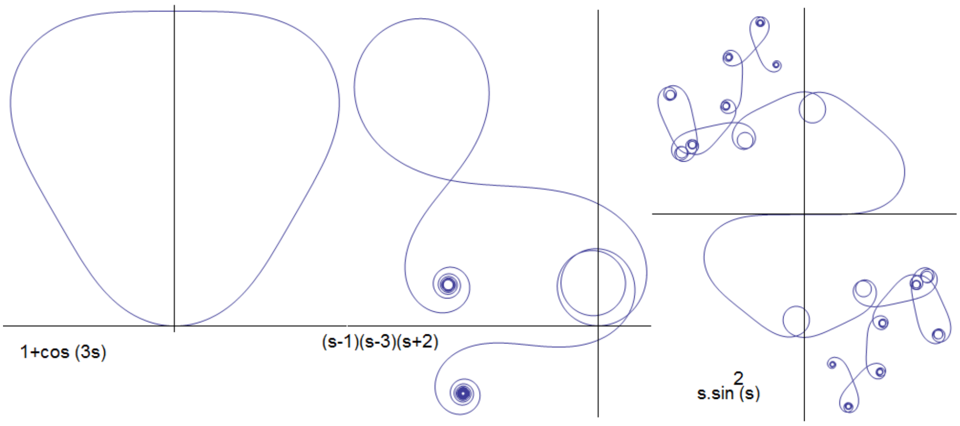


Figure 1: Curvature $\kappa(s) = 1 + \cos(3s)$, $(s-1)(s-3)(s+2)$ (each zero gives an inflexion point) and $s.\sin^2 s$ from s= - 30 to s=+ 30.

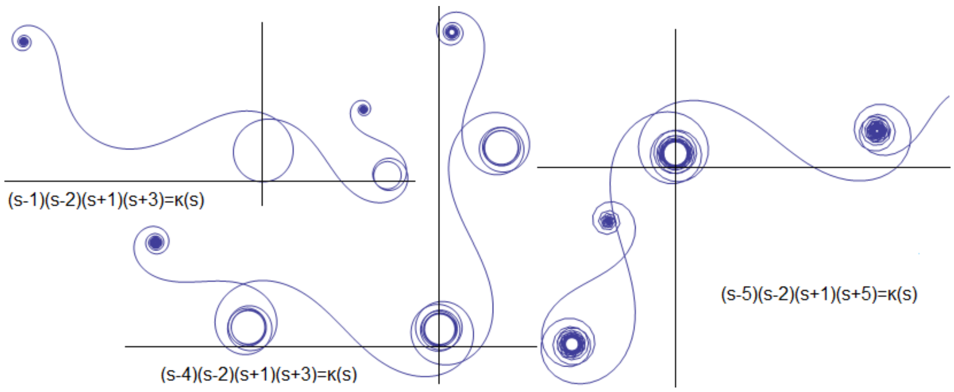


Figure 2: Curvature $\kappa(s) = (s-1)(s-2)(s+1)(s+3)$, $(s-4)(s-2)(s+1)(s+3)$ (symmetric) and $(s-5)(s-2)(s+1)(s+5)$.

# 2 Parametric equations, arc length s and curvatures.

## 2.1 Intrinsic curves in arc length : $R_c = f(s)$

The above Cesaro or Whewell equations suggest to use arbitrary functions of a parameter u to create examples of curves defined by intrinsic equations. The two points of view : radius of curvature or curvature which are inverse fonctions and difference can be strengthened in the following way :

**Curvature:** $\kappa(s) = 1/R_c$ is a kind of internal pilot that fixes the angle direction of the trajectory. The integral of the curvature along the arc length $\int \kappa(s).ds$ is Whewell angle at s.

**Radius of curvature :** $R_c(s)$ determines the curvature angle of the curve by comparing it to the circle and it is the point of view of Gregory's transformation.

### 2.1.1 Arc length element defined by an arbitrary function $f(u)$

Suppose $ds = f(u).du$ and V=u are given then the parametric equations of the curve are :

$$x - x_o = \int_{u_o}^{u} f(u).\sin u.du \qquad y - y_o = \int_{u_o}^{u} f(u).\cos u.du$$

$$s - s_o = \int_{u_o}^{u} f(u).du \qquad \tan V = \frac{dx}{dy} = \tan u \qquad V = u$$

So this comes down in using V (Whewell angle) as the parameter and as we have seen in Gregory's transformation (Part I). V is a natural parameter for curves in the plane.

### 2.1.2 Angle V defined by an arbitrary function $g(u)$

If we suppose that V= g(u) and f(u)=1 are given then the parametric equations of the curve are :

$$x - x_o = \int_{u_o}^{u} \sin[g(u)].du \qquad y - y_o = \int_{u_o}^{u} \cos[g(u)].du$$

$$s - s_o = \int_{u_o}^{u} du = u - u_o \qquad \tan V = \frac{dx}{dy} = \tan[g(u)] \qquad V = g(u)$$

So this is just using the arc length s as the parameter to define the curve in the plane.

### 2.1.3 Using simultaneously two arbitrary functions $V = g(u)$ and $ds = f(u).du$

We can combine the two ways in a more complete form and write :

$$x - x_o = \int_{u_o}^{u} f(u).\sin[g(u)].du \qquad y - y_o = \int_{u_o}^{u} f(u).\cos[g(u)].du$$

$$s - s_o = \int_{u_o}^{u} f(u).du \qquad \tan V = \frac{dx}{dy} = \tan[g(u)] \qquad V = g(u)$$

$$z(M) = \int_{u_o}^{u} \rho(u).e^{i.\theta(u)} du = \int_{u_o}^{u} f(u).e^{i.g(u)}.du = \int_{s_o}^{s} e^{i.V(s)}.ds$$

**Example of Si-Ci Spiral** This spiral is defined by $g(u) = u$ and $ds = \frac{du}{u}$:

$$x - x_o = \int_{u_o}^{u} \frac{1}{u}.\sin u.du \qquad y - y_o = \int_{u_o}^{u} \frac{1}{u}.\cos u.du$$

$$s - s_o = \int_{u_o}^{u} \frac{du}{u} = \ln(u/u_o) \qquad \tan V = \frac{dx}{dy} = \tan u \qquad V = u$$

And the Cesaro intrinsic equation is : $R_c = 1/u = e^{-s}$

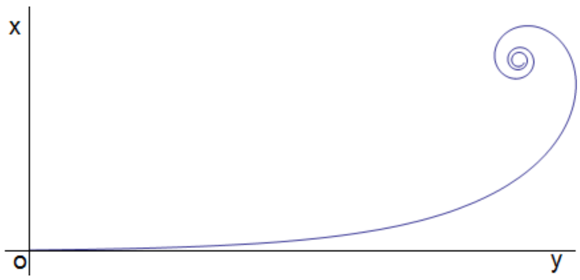


Figure 3: Curvature $\kappa(s) = \frac{1}{R_c} = e^s$

# 3 Curves with periodic curvature : $\kappa(s) = 1/R_c$ function of s with period T.

## 3.1 Geometric interpetation of closure condition for a plane curve with T-periodic curvature $\kappa(s)$

In paper (D) is given the condition on the periodic (T) intrinsic equation $\kappa(s) = 1/R_c(s)$ so to obtain a closed plane curve:

$$\frac{1}{2\pi} \int_{0}^{T} \kappa(s)ds = \frac{m}{n} \in \text{Q - Z} \qquad [1]$$

There is a kind of equivalence of this proposition with the requisition in Gregory's transformation to get a closed curve for the wheel rolling on a given plane curve. Inverse Gregory's transformation (see Part II and V) gives an integral condition on the angle after one (or h integer) turn(s) of the closed wheel or after one (or h integer) period (s) of the ground. The above condition proved in (D) implies that the resulting curve is closed and in the special cases $\frac{m}{n} \in Z$ the resulting curve is periodic (along a fixed direction). This is the case when angle after an integer number of periods is a multiple of $2.\pi$. So we can add periodic curves (along a line) when we use Gregory's transformations ($TG/TG^{-1}$). Or simply consider periodic curves (in a direction) as special closed curves that close only at $\infty$.

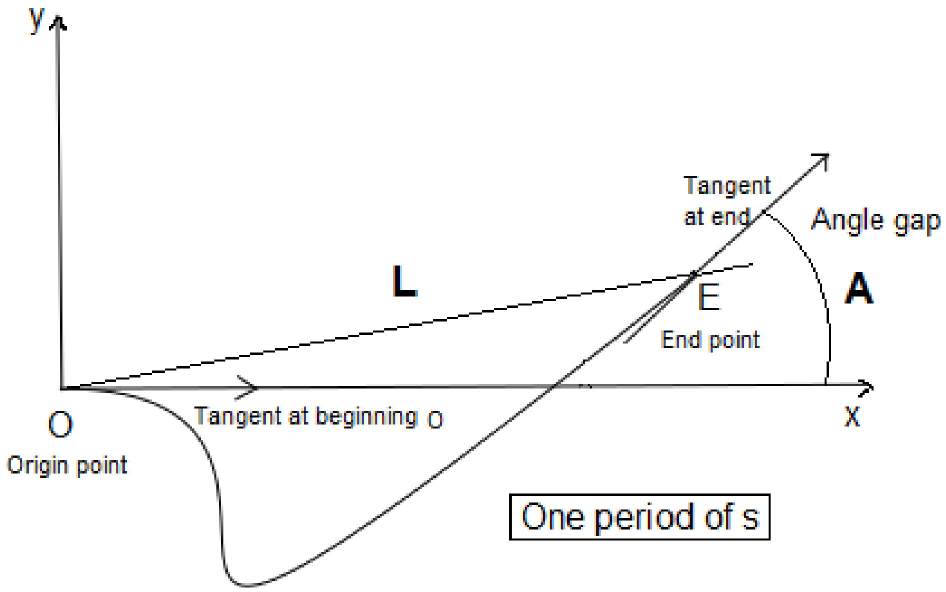


Figure 4: One period partial graph of $R = f(s)$ from s=0 to s=T.

### 3.2 Closed or periodic curves in the plane.

The fact that $\kappa(s)$ - and so $R_c$ - is a T-periodic function has a geometric interpretation : each part (segment of the curve) corresponding to one period in s is identical and between the tangent at the ends are the same angle A. We trace a part of the curve beginning at the origin and tangent at O to the x-axis and create the curve progressively beginning with one period T of the curvature. At the end point $E(x, y)$ the oriented tangent is in the direction A w.r.t. the x-axis. This segment of the curve will repeat rotated of A infinitely. And the only important variable is the angle gap A after exactly one period. The length L=OE only fixes the scale of the curve.

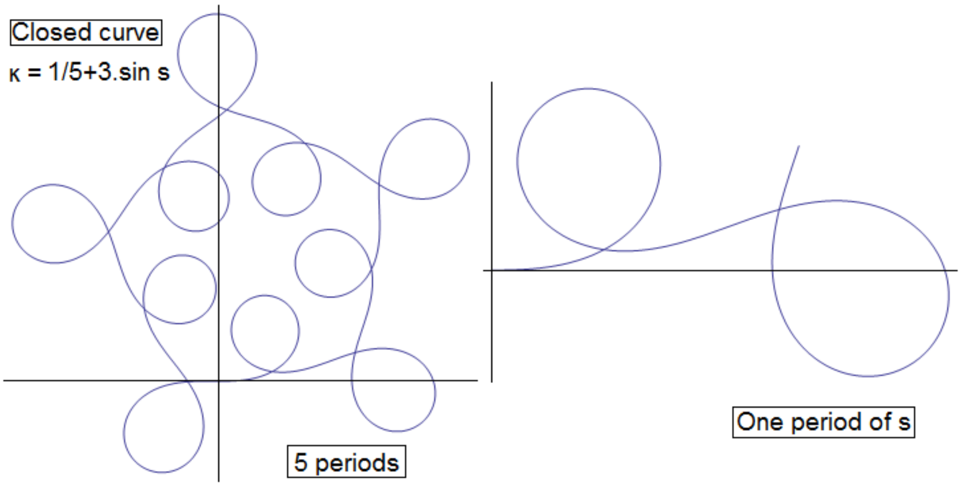


Figure 5: Curvature $\kappa(s) = 1/5 + 3\sin(s)$ is a closed curve

The condition for the closure is : the polygone with constant side OE and angle between two consecutive sides A will close or not as this angle is or is not commensurable to $2\pi$. If this angle is zero then we obtain the special case of periodicity along a line or simply T-periodic plane curves.

### 3.3 Analogy with regular polygons inscribed in a circle

We have mentioned in the last section the similarity with polygonal lines inscribed in a circle that close only if the angle at the center for a side is commensurable to $2.\pi$. The angle from O between the position at the beginning and at the end of a period of curvature is in proportion $\frac{m}{n}.2\pi$ with $m/n \in Q - Z$. In our explanation we replaced the segment of curve by an equivalent segment of line and an angular gap.
Gregory's transformation leads to a similar result (see parts II and V) and same conclusion for closed or periodic (along x-axis if $\frac{m}{n}.2\pi$ with $m/n \in Z$) curves.

## 4 Some examples of curves defined by Cesaro intrinsic equation : $\rho = f(s)$

### 4.1 The wheel corresponding to a circle-ground w.r.t. tangent

The curve (see Part III) has for parametric equations :

$$C_{-2}(1,0) \rightarrow \rho = \cos^2 u \qquad \theta = \tan u - 2u$$

It has a loop and an asymptotic point at the pole. It is the inverted of Sturm/Norwich spiral. Its length is s=u and intrinsic equation is $R_c = 1/(3 - \tan^2 s)$.

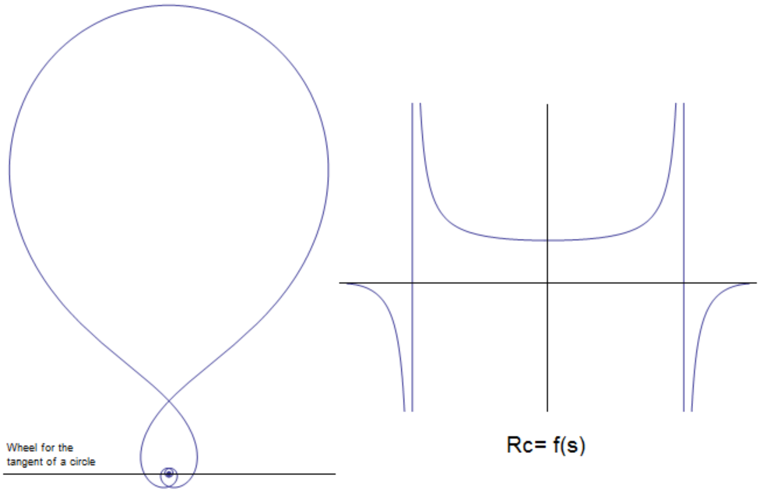


Figure 6: Wheel for a circle and its tangent / Radius of curvature = $f(s)$. The inflexion points correspond to the two asymptotes

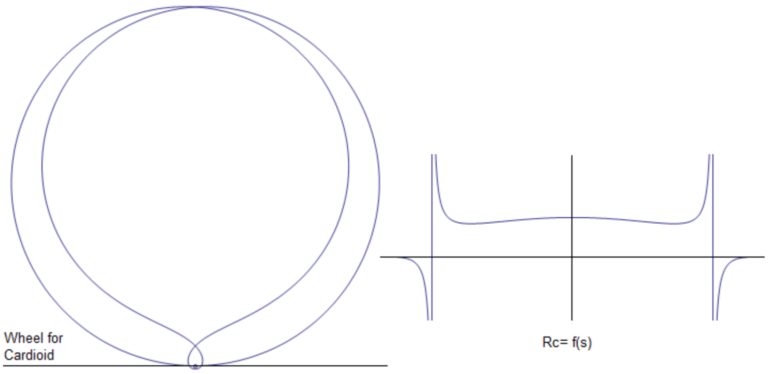


Figure 7: Wheel for Cardioid and axis $f(R_c, s) = 0$

### 4.2 The wheel corresponding to a cardioid-ground w.r.t. to its symmetry axis

The curve (see Part III) has for parametric equations :

$$C_3(1, 1) \rightarrow \rho(u) = \cos^2 u.\sin 2u \ , \ \theta = \tan u - 4u$$

It is a closed curve and its total length is 4, arc is $s = 2\sin u$. Its intrinsic equation is (see Part III) :

$$R_c^2.[7s^2 - 24]^2 = [4 - s^2]^3$$

Ther are two inflexion points ($Rc = \pm\infty$).

# 5 Integrals $\pi = \int_{-\infty}^{+\infty} \kappa(s).ds$ and Cesaro intrinsic equation : $R_c = f(s)$

The integral of the curvature along the arc length gives the global angle of rotation of the unit tangent. This is a kind of measure of the number of foldings of the curve. An integral from $-\infty$ to $+\infty$ with value $\pi$ gives an example of curvature for a curve of infinite length and indice of rotation of the tangent equal to $k.\pi$. We present in this section two examples with known intrinsic equations :
1- General intrinsic equations of Alysoids is : $k.R_c = 1 + s^2$ and
2- Curves verifying : $R_c = \frac{1}{k}\cosh(s)$. This class has not known name. For k=1 or 2 in the two cases :

$$\int_{-\infty}^{+\infty} \kappa(s).ds = \pi \text{ or } 2.\pi$$

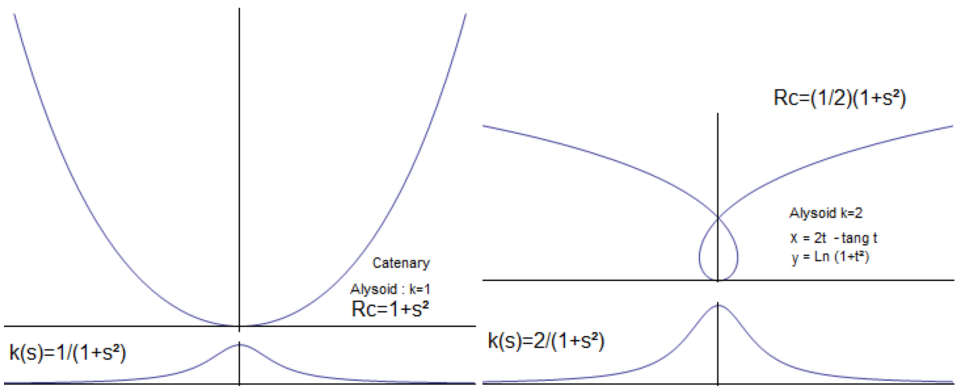


Figure 8: Curvature $\kappa(s) = \frac{k}{1+s^2}$ k=1, 2

## 5.1 Example 1 - Alysoids : the Catenary (k=1) and curve k=2.

The Catenary is $y = \cosh(x)$ is the Alysoid for k=1 and $R_c = 1 + s^2$. The Alysoid for k= 2 has for parametric equations in an orthonormal frame :

$$x = \log(1 + \tan^2 u) \qquad y = 2.u - \tan u$$

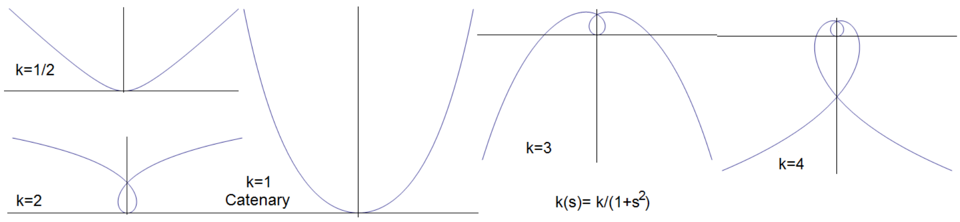


Figure 9: Curvature $\kappa(s) = \frac{k}{1+s^2}$ for k=1/2, 1, 2, 3, 4

The intrinsic equation is $\frac{1}{R_c} = \kappa(s) = \frac{2}{(1+s^2)}$. The graphs of the cuvature $\kappa(s)$ of the two curves are represented with the resulting curve and shows a maximum of curvature at the summit. The axial symmetry for the graph of curvature implies a symmetry for the curve itself. Since $\pi = \int_{-\infty}^{+\infty} \kappa(s).ds$ for the Catenary the tangent turns of global angle $= \pi$ after having described all the curve. For the other curve (k=2) the global angle is double so $= 2\pi$ and this comes down to a folding of $\pm\pi/2$ for each of the two infinite branches of the curve and the creation of a loop.

### 5.2 Example 2 : Poleni's curve and catenary of uniform strength.

The Catenary of uniform strength has for equation : $e^{-\frac{y}{a}} = \cos\frac{x}{a}$ and Cesaro equation $R_c = \cosh s$.
Poleni's curve (sometimes called "Forcats" curve) has for parametric equations in an orthonormal frame :

$$x = u - 2.\tanh u \qquad y = \frac{2}{\cosh u}$$

Its Cesaro equation is : $R_c = \frac{1}{2}\cosh(s)$ so its half the one of the Catenary of Uniform Strength.

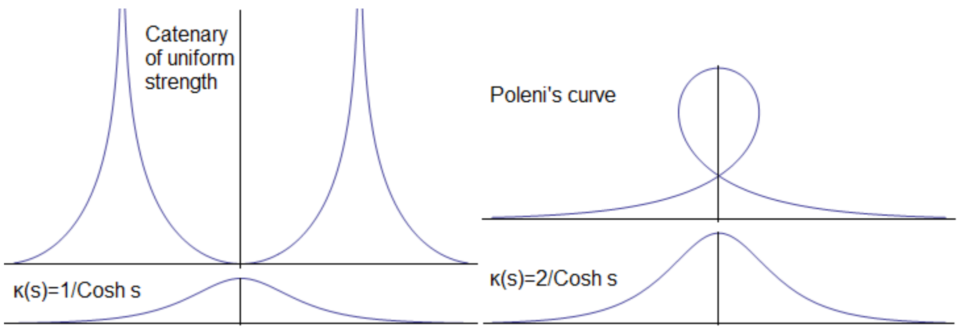


Figure 10: Curvature $\kappa(s) = \frac{k}{\cosh(s)}$ k=1, 2.

The graph of the cuvature $\kappa(s)$ is represented with the resulting curve. We have $\int_{-\infty}^{+\infty} \kappa(s).ds = \pi$ for one arch of the Catenary of uniform strength the tangent turns of global angle $= \pi$. This angle is $2\pi$ for Poleni's curve so there is a loop.

## 6 The successive antipedals of the point

In Part IV Tschirnhausen's cubic (TC) we mentioned the three first antipedals of the point: line, parabol and TC.These are sinusoidal spirals $\rho = \cos^{-n}\frac{\theta}{n}$ which present 0 folding for the line (n=1), 1 for the parabola (n=2) and 2 for the TC (n=3) and 3 for the the following one (n=4). Antipedal transformation add an angle $\pi/2$ for each parabolic branch of the curves. So it is very similar to the foldings explained in the previous section. For the sinusoidal spirals of equation $\rho = \cos^{-n}\frac{\theta}{n}$ the pole is a focus of the curve.

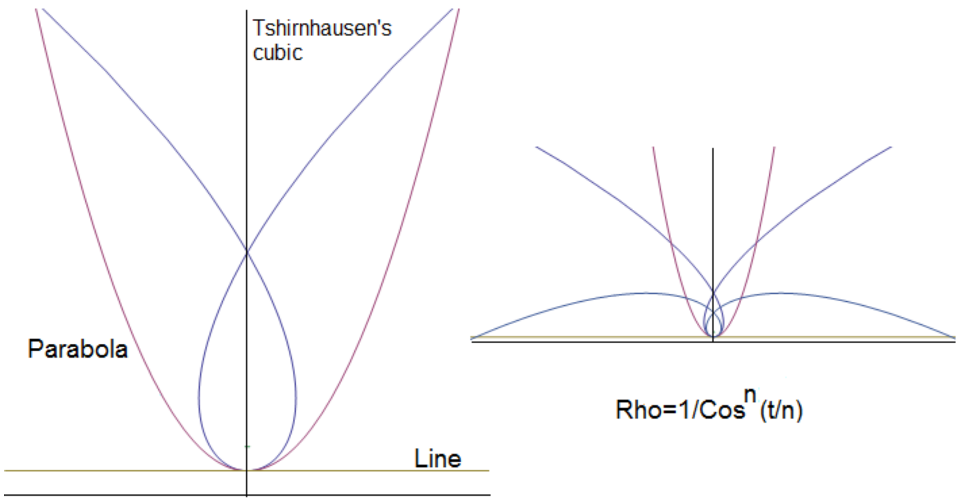


Figure 11: 4 antipedal of the point : Line : 1, Parabola : 2, TC : 3 and 4

## 7 Methods to obtain the closure : Adding a constant or Scaling the curvature function

We return to the curves with periodic curvatures (section 3 above). Paper [D] gives two methods to modify the equation so that we obtain from a not closed curve a closed curve. The first is simply adding to the curvature function an adaptated constant in $k_a(s) = k(s) + a$. The second, the scaling, consists in multipliying $\kappa(s)$ by a well chosen constant b : $k_b(s) = b.k(s)$.And the integral condition (1) of section 3 and [D] is the mean to compute the

constant or scale. The geometrical point of view of Gregory's transformation (see part I) gives also a methods to construct closed curves in a similar way by a continuity argument. (see part I - 1.1.2) : an adaptated angle dilation h $[\rho = y, \theta \to h.\theta \quad h \in \mathbb{R}]$ of the wheel which is equivalent by GT to an affinity along the base-line (x'x axis) or a well chosen y-affinity of the ground can also lead to a closed wheel.
With Gregory's transformation we use radius of curvature instead of curvature and there is other ways to get the closure : move in parallel the base-line of the adjusted distance, add a constant length to $\rho$ (conchoids), etc.

## 8 Closed curves and periodic curves

In parts II and V we have considered the same ground and base-lines moved in parallel. This gave a mean to find closed curves as wheels for the same ground for special cases as a circle (closed) and a cycloid (periodic). It seems that there is in the plane a connection between periodic and closed curves since the latter is simply a special case with P=0 (no gap after k rotations). It is in this sens natural to place in the same class periodic curves (in a direction) and closed curves. And we could consider also a rotation periodicity around a fixed pole. The curve is not necessary closed as we have seen above. If the rotation center (center of the polygon) goes away in the infinity then we have the classical periodicity along a line.

## 9 A remark about the foci of algebraic plane curves

Foci of algebraic plane curves seem to have a relation with Gregory's transformation and the corresponding geometric view would permit to give a new conception of these special points known since antiquity for the conics (Apollonius). The focus of Euler Serret curves (see Part II) are situated inside the smallest loop and is the pole of the (algebraic) closed wheel. This observation could bring an alternate approach to the usual definition of these foci based on notions coming from the complexified plane and cyclic points (ombilics): they are circles of null radius bi-tangent to the plane curve or equivalently the points from which we can lead two tangents through the two circular points at infinity I, J described by the homogeneous coordinates (1, i, 0) and (1, -i, 0) to the algebraic curve. It might be possible to define the foci in terms of folding/distortion of a line from these foci by analogy with point of attachment used in computer-aided design. If these foci are chosen as the pole of wheels that may simplify computations. The foci play an important role in the geometry of algebraic plane curves and should be defined by a real geometric argument like for conics. The study of algebraic plane curves by Gregory's transformation could bring a new definition of the foci of an algebraic plane curve.

## Annexe I : Roulette, Pedal and Gregory's transformations.

Parametric equations of the roulette $(x, y)$ of the pedal $(\rho, \theta)$ and Gregory transformations $(GT, GT^{-1})$
Roulette : the formulas to determine the cartesian parametric equation $(x, y)$ of the roulette on the line xx' of a curve given in polar coordinates $(\rho, \theta)$ :

$$y = \rho \sin(\theta) \text{ and } x = \int_{\theta 0}^{\theta 1} ds_w - \rho \cos(V)$$

Pedal : the formula to obtain the parametric polar equation of the pedal of the polar curve $(\rho_0, \theta_0)$ of a curve given in polar coordinates $(\rho_1, \theta_1$ is :

$$\rho_1 = \rho_0 \sin(V) \text{ and } \theta_1 = \theta_0 - (V - \pi/2)$$

The equation of $n^{th}$ pedal is $(n \in \mathbb{Z})$ :

$$\rho_n = \rho_0 \sin^n(V) \text{ and } \theta_n = \theta_0 - n(V - \pi/2)$$

Gregory : The formulas associated to direct $(GT)$ and inverse $(GT^{-1})$ Gregory's transformation from the wheel to the ground, direct Gregory's Transform is :

$$y = \rho \qquad \text{and} \qquad x = \int \rho.d\theta$$

in the opposite way $GT^{-1}$ from the ground to the wheel :

$$\rho = y \quad \text{and} \quad \theta = \int \frac{dx}{y}$$

$$\tan V = \frac{\rho.d\theta}{d\rho} = \frac{dx}{dy}$$

The $GT^{-1}$ is defined in the whole euclidean plane except on the line $y = 0$ for which the angle of the wheel is not defined. This line is called the "base-line"and is the dual of the pole. A translation in the ground-plane corresponds to a rotation in the wheel-plane.

This article is the $X^{th}$ part on a total of 10 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length

as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta$-curves.
Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, closed or periodic curves and Gregory's transformation.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr

References :

A - H. Brocard , T. Lemoine - Courbes geometriques remarquables Blanchard Paris 1967
B - Nouvelles annales de mathematiques (1842-1927) Archives Gallica
C - Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
D - J. Arroyo, O.J. Garay, J.J. Mencia - When is a periodic function the curvature of a closed plane curve - American Mathematical Monthly (may 2008).
E - Antonin Slavik - Intrinsically defined curves and special functions. Mathematical Magazine 86 (June 2013).
F - Matthias Kawski - Curvature for everyone - 8th Asian Technology Conf Math, Hsin-Chu, Taiwan 2003.
G - http://www.mathcurve.com/courbes2d/syntractrice/syntractrice.shtml