Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates - Part XI -

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Abstract

We present a natural correlative of the inversion (transformation by reciproqual radii) in 2-dimensional euclidean space : the Transformation par Semi Droites Reciproques proposed by E. Laguerre about 1880 and give some elementary properties. This TSDR is a tangential version of the inversion and shares with it many correlative caracteristics. Two systems of coordinates in the plane are described : polar tangential coordinates which goes back to Euler and axial coordinates (λ, θ) exposed by M.d'Ocagne (1884). We give examples of tangential axial transformations.

1 The inversion :

1.1 Definition of inversion.

Inversion as a geometric transformation was first described in a paper of J.B. Durrande (1798-1827) in Annales de Gergonne 11, pp 1-67(1820-21) published in july 1, 1821 (11). It was called transformation by reciproqual radii. If we limit the definition to the plane and to real objects, it consists in a point O, the center of inversion and the relations between two corresponding points M and M' is :
1 - M, M' are colinear with point O (center of inversion)
2 - The product of lengths OM . OM' = k ($k \in R$ is constant). In polar coordinates centered in O we have :

$$\boxed{\rho\ .\ \rho' = k}$$

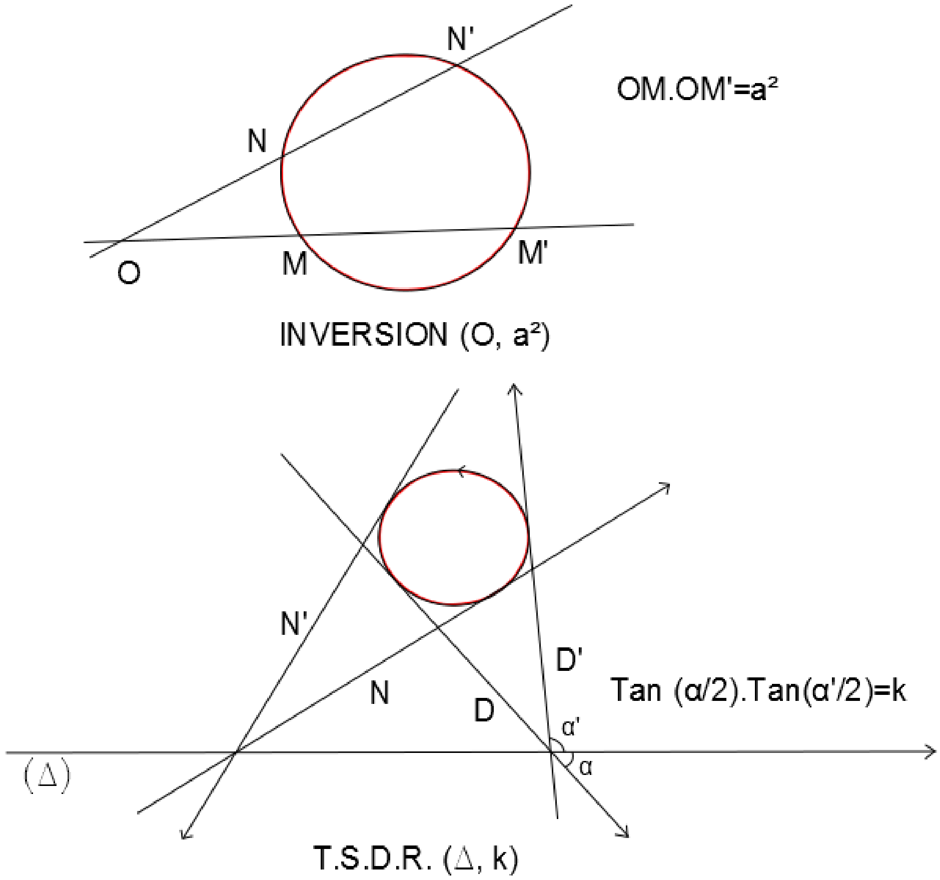

Figure 1: Inversion : 4 points and TSDR : 4 lines

1.2 Some properties of the inversion.

Two given corresponding points M and M' colinear with a center O define globally the inversion. Take any point N in the plane ($N \neq O$) then the point N' transformed of N is the second point of intersection of the circle (or line) circumscribed to triangle (MM'N) with the line through O and N. So we have ON . ON' = k.
A circle or a line is transformed in a circle or in a line if the circle or the line passes through O.
Inversion preserves the angles ($V \to -V$) so is a conformal transformation.
If $k > 0$ then there is a real circle of fixed points. $\rho = \sqrt{k}$
If $k = 0$ then all points are collapse on O.
If $k < 0$ then there are no double point. Inversion has known many generalisations, it is a special conformal representation in the plane $Z = f(z)$ and an example of bi-rational map (algebraic geometry). Poincare has used it to describe his Disk or Half plane and a model of hyperbolic geometry.

Anallagmatic geometry that preserves circles, spheres and hyper-spheres has been generalized to n-dimension spaces. By a theorem of Liouville this conformal group is the unique case for $n > 2$ and the plane for n=2 is special from this point of view.

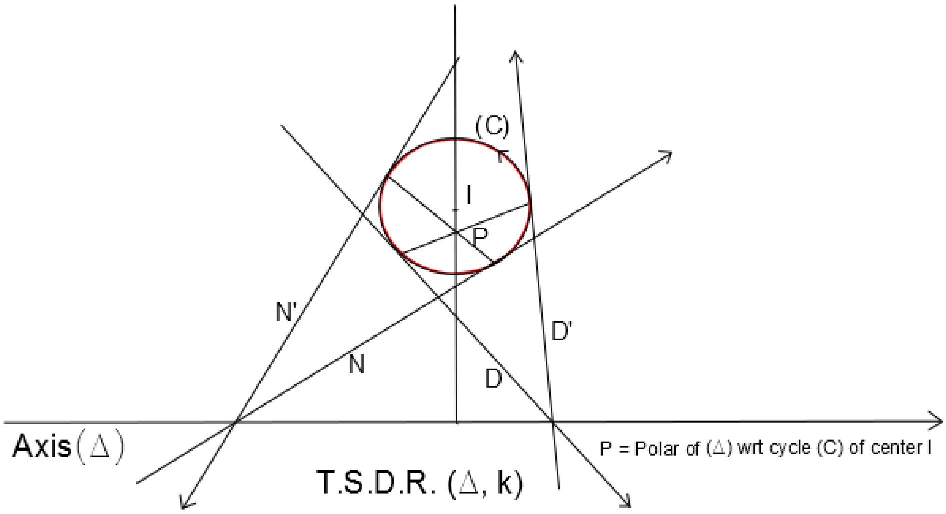

Figure 2: TSDR: T(D)=(D') , T(N)= (N') with cycle (C) tangent to (D), (D') and (N) : (N') is the second tangent.

2 The "Transformation par Semi Droites Reciproques" of Laguerre (1880-83).

2.1 Oriented cycles and oriented lines in the plane: "Geometry of Direction".

Laguerre gave orientation to objects in geometry as plane curves when it is possible : the line and the circle can easily support a direction. Just put an arrow on it and it is a Cycle or a Semi Line. The same circle with each of the two possible orientations are different cycles. We can choose the radius R of the circle as positive, $R \geq 0$, same as angles in the trigonometric cycle, and negative if clockwise with $R \leq 0$. A cycle of radius $R = 0$ is a point and has indeterminate orientation.The Laguerre-geometry deals with cycles and semi-lines with orientations. This precision makes a real progress in the definition of objects since tangency is between two objects with same point of contact and same orientation at this point. Three semi-lines have a unique inscribed cycle instead of 4 circles. Two cycles have a unique center of homothety. There is at most two cycles tangent to three given cycles in the plane instead of eight if we do not take into account the orientation. The Transformation par Semi Droites Reciproques will be named TSDR fort short from now on.

2.2 The T.S.D.R. an axial transformation correlative of the inversion.

This plane transformation is defined by analogy to the inversion : the correspondance between points of the inversion is translated in a correlative correspondance between semi-lines. A special line is called the axis of the TSDR (correlative of the center of inversion). To each element or property of the inversion there exists a correlative element or property of the TSDR. The geometric objects are oriented (semi-lines, cycles). M. D'Ocagne distinguishes central transformations which have a special central point (homothetie, inversion, rotations, similarities...) and axial transformations (that we examine in this paper) and which have special semi-line and a correlative interpretation. T.S.D.R., as a geometric transformation, was first described in a paper of E. Laguerre (1835-1885) in the Nouvelle Annales de Mathematiques "Sur la Geometrie de direction Seance du 4 et du 18/06/1880). Laguerre writes : "The TSDR seems to me, in the study of plane geometry, to have to be used advantageously beside the inversion". First he gave the name "Transformation par Directions Reciproques" but later modified the apellation to "Semi Droites Reciproques" to include the orientation in the definition. If we limit to the plane and to real objects, it consists in a line, the axis of of the TSDR and the requisitions for correspondance between two semi-lines D and D' are :
1 - Semi-lines D, D' intersection is on the oriented axis (Δ) (axis of the TSDR)
2 - The angles α and α' of semi-lines D and D' with oriented axis (Δ) verify the relation (with k $\in R$ a constant). :

$$\boxed{\tan(\alpha/2).\tan(\alpha'/2) = k}$$

2.3 Some properties of the T.S.D.R.

Given two corresponding oriented semi-lines D and D' that cut at T on oriented axis Δ defines globally the TSDR. Take any semi-line N in the plane ($N \neq \Delta$) then the semi-line N' transformed of N is the second semi-line tangent to the unique oriented cycle inscribed in (D, D', N) from the point T' where semi-line N cuts the axis Δ. It can be proven by elementary geometric arguments using the pole of Δ that we have $\tan(\alpha/2).\tan(\alpha'/2) = k$.
A cycle or a semi-line is transformed in a cycle or in a semi-line if the semi-line is not the axis Δ.

Centers of cuvature of two transformed curves by TSDR at corresponding

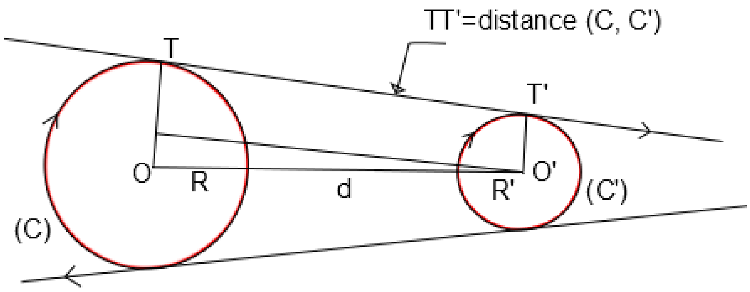

Figure 3: Distance between oriented cycles

points are on a perpendicular to the axis. A circle is transformed by TSDR in another circle with radical axis the one of the TSDR.

2.4 Distance between cycles in the plane

Laguerre gives the following definition of this distance : it is the length of the segment measured on one of the two tangents to the cycles. It is a positive number. This distance is equal to $T^2 = d^2 - (R - R')^2$ with d the euclidean distance between the centers. For example the distance between two opposite cycles (same center and radius $R' = -R$) is $T^2 = -4R^2$. The distance T^2 between cycles is preserved by the TSDR. This is the correlative of the well known property of inversion that preserves angles (Laguerre).

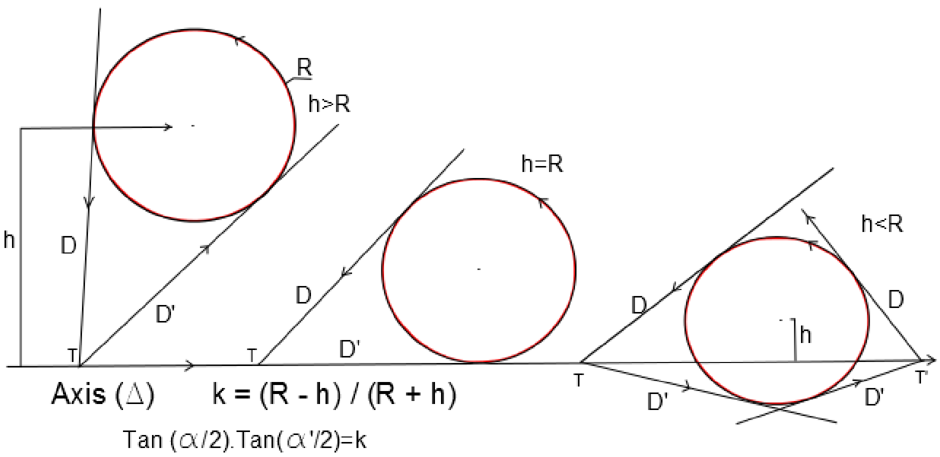

Figure 4: 3 kinds of TSDR

2.5 Classification of TSDR.

The number k ($\in R$) defines the special TSDR with axis Δ and is only function of the ratio h/R. All the cycles (C) used to construct the transformed semi-line are homothetic with centers on Δ and ratio h/R. We have :

$$\boxed{k = \frac{R-h}{R+h} = \tan(\alpha/2).\tan(\alpha'/2}$$

We have three different cases if $h > R$ the cycles doesn't cross the Δ-axis, if $h = R$ then the cycles are tangent to the Δ-axis and if $h < R$ then the cycles cut the Δ-axis at two points. In the first case there are no real double semi-lines, in the second case the Δ-axis is the only double semi-line and in the third case there are two double directions (parallele semi-lines) given by $\tan^2(\alpha/2) = k$. The analogy with linear families of cycles and radical axis of Poncelet (with two base points, tangent to a line at a point, with two limit points). These 3 cases correspond to the values of k :

If $k < 0$ then there is no real double semi-line.
If $k = 0$ then is only one semi-line the Δ-axis itself.
If $k > 0$ then there are two double semi-lines. They are the double tangents to a pair of corresponding cycles by the TSDR.
For cycles transformed by TSDR the double semi-lines are given by the double tangent of the inital cycle and its cycle image by TSDR.

2.6 Important cases of TSDR.

In his paper of 1880 Laguerre indicates two particular cases :
1- Double directions are opposit semi-lines then the TSDR is the axial symetry wrt the Δ-axis.
2- Doubles directions are isotrope semi-lines with intersection O. Then the TSDR is the central symetry wrt to the center O.

2.7 Some properties of the TSDR after Cesaro (8).

1- If M is the current point of (C) and T is the intersection of the tangent with the Δ-axis the circle of center T through M is the locus of all different axial-TSDR with various k corresponding to M. Tangent lines at all curves cut the Δ-axis at the same point T and have equal length between T and contact point M. This is the correlative of the fact that all inverses M' of a given point M is transformed in the line through O and M.
2- The center of curvature of two corresponding points are on the same orthogonal line to the Δ-axis at T.

INVERSION :	**T.S.D.R.:**
1. Center of inversion O	1. Axis of TSDR : (Δ).
2. Lines through O	2. Semi-lines secant on (Δ).
3. Length from O to M	3. Angle of semi line with Axis of TSDR : (Δ)
4. Product of length OM. OM'= k (relation between distances)	4. Product of $\tan(\alpha/2)$ of (1/2) angle (Semi-line M, Semi-lineΔ) and (1/2)(Semi LineM', Semi line Δ). (relation between angles)
5. Angle V at intersection of curves.	5. Distance between tangent points on common tangent of curves
6. Relation independant of angle direction around O.	6. Relation independant of place of secant point on (Δ)

Table 1: Correlative elements and caracteristics of Inversion and TSDR

Corollary : a circle transformed by TSDR wrt to any Δ-axis is another circle. This line is the radical-axis of the two corresponding circles.
3- The tangential distance of two curves is equal to the tangential distance between curves transformed by TSDR. This is the correlative translation of the property of inversion preservation of angles at curves intersections.
4- Among the curves tranformed of a given curves there are two at each time that have a cusp. These cusps are on the the circle (T) - locus of contact points - cut the perpendicular to the Δ-axis - locus of center of curvature. The locus of cusps is the envelope of (T).

2.8 A generalisation of polarity for a circle : Hypercycles.

Laguerre has generalised the theory of polarity wrt a circle in the plane (reciproqual polarity) equivalent to the composition of an inversion and a pedal (or antipedal) transformation. Instead of the correspondance between a point and a line (polarity), he defines a new curve analog to the cycle but of class four that he calls hypercycle created by the mean of a tangential axial correspondance (harmonic system) between two cycles or semi-lines. Then he defines an harmonic system of four semi-lines - lines with direction - as a set of two couples of semi-lines (A, A') and (B, B') that are tangent

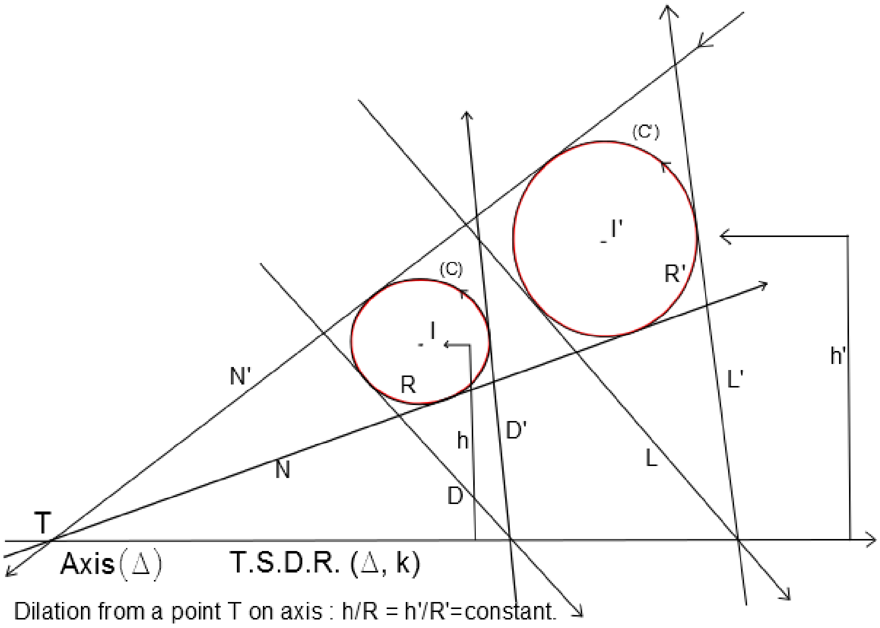

Figure 5: Homothetic constructions of semi lines for a positive TSDR ($k < 0$)

to a same cycle and their contact points divise harmonically the cycle; A' is the harmonic conjugate of A wrt the couple of semi-lines (B, B').
A fondamental property is that TSDR preserves the hypercycle as well as the cycles.

The hypercycle of Laguerre is a curve of class four defined by the following property : harmonic conjugates of a semi-line in the plane wrt couples of conjugate tangents to the hypercycle have for envelope a circle K. In an hypercycle there are two fixed semi-lines called fondamental semi-lines and tangents can be associated in couples (conjugate tangents) if they form a harmonic system with the fondamental semi-lines. A same hypercycle can be defined in an infinity number of ways.
But Laguerre never gives a figure to show how all this works and to my (unfortunately limited) knowledge I could not find a paper or a picture of what looks like one of these hypercycles. They are curves of direction and he enumerates some exemples : astroid, double-parabola, their parallele curves and more generally all catacaustics of the parabola for parallele incident light rays.
Harmonic conjugates of a given fixed semi-line Δ in the plane wrt all the couples of conjugate tangents of a hypercycle have for envelope a cycle Π. This cycle Π is the polar cycle of the semi-line Δ. Let us quote Laguerre : "Properties of these cycles present an analogy with properties of poles and lines in the theory of conics".

Here are some properties (Laguerre 1882) :
1- If a semi-line Δ has for polar cycle Π, the polar cycles of all tangents to Π are tangent to Δ.
2- The polar cycle of a tangent to the hypercycle touches the curve at the point of contact of the tangent.
3- If we call Π an Π' the polar cycles of two semi-lines Δ and Δ' the polar cycles of common tangents to Π and Π' touch Δ andΔ'.
4- If two semi-lines Δ and Δ' are opposit, common tangents to Π and Π' have for polar cycle two points. There are an infinity of semi-lines that reduce to points and the locus of these points is a conic.

An hypercycle is defined if we know the two fondamental semi-lines and the polar cycle of a given semi-line in the plane, so we can construct all the tangent we need.

Hypercycles are, in the general case, curves of class 4 and order 6, with three double tangents and one of those is the line at infinity passing though cyclic points. They are catacaustics by reflection of conics for parallele incident rays. The foci of the initial conic are foci of the hypercyle.

When this conic is a parabola Laguerre gives to these catcaustics the name of cubic hypercycle which has special properties. A cubic hypercycle has a unique singular focus (the same one as the parabola) and the sum of distance from the focus to any conjugate tangents is constant.
The cubic hypercycles are of class 3, order 4, with one apparent double-tangent, passing through the cyclic points and tangent to the line at infinity. They are involutes of the Tschirnhausen's cubic or anticausics of the parabola for parallele light rays coming from any direction.

A general hypercycle is the transformed by TSDR of a double-parabola.
This double-parbole, superposition of two parabolas with opposit directions, is an ideal case : the double-axis (2 semi-lines with opposit directions) of the parabola is the couple of fondamental semi-lines and two conjugate tangents are symmetric wrt the axis of the parabola.

3 Euler Polar Tangential Coordinates (EPTC).

Euler used a tangential sytem to study curves in the plane by mean of a normal equation of the tangent line :

$$\boxed{x.\cos(\theta) + y.\sin(\theta) - p(\theta) = 0} \qquad [1]$$

The angle θ is the parameter and (x, y) are the cartesian coordinates. The coordinates of the point where this line is in contact with the envelope is given by :

$$\boxed{x = p(\theta)\cos\theta - p'_\theta \sin\theta \quad \text{and} \quad y = p(\theta)\sin\theta + p'_\theta \cos\theta \quad [2]}$$

The curve in polar coordinates $\rho = p(\theta)$ is the pedal from O of the envelope of the line [1] so this problem is equivalent to find the anti-pedal of a curve C in polar coordinates $\rho = p(\theta)$.

3.1 Unicursal curves with rational arc length :

Algebraic curves given by rational coordinates in homogenous coordinates by polynomials in t : x= f(t), y=g(t) and z= h(t) or x(t) = f(t)/h(t) and y=g(t)/h(t) as rational cartesian coordinates. In general these curves have not a rational arc length (without radicals) but EPTC is a mean to obtain curves with a rational arc. J. Haag proved in 1915 (10) that :
If ϕ and p are two any rational functions of t ϕ' and p' their derivatives wrt t. Then the envelope of (C) of the line :

$$\boxed{\tfrac{1-\phi^2}{1+\phi^2}.x + \tfrac{2.\phi}{1+\phi^2}.y = \tfrac{1+\phi^2}{2}.\tfrac{p'}{\phi'}}$$

gives the most general curves with rational cartesian coordinates and rational arc length (functions of t). The value ϕ is equal to $\tan(\frac{\theta}{2})$, with the θ of Euler tangential equation.

$$s = -(p + \frac{d^2p}{d\theta^2})$$

This provides a method to find as many curves with rational coordinates and rational arc length at will.

4 Axial Coordinates (AC) of M. D'Ocagne

In an article of NAM (december 1885) d'Ocagne proposed a system of tangential coordinates : a given line of the plane is the axis with origin O. At distance lambda from O the line cut the axis with oriented angle theta counted with respect to the axis. A relation between lambda and theta : $f(\lambda, \theta) = 0$ or a common parameter t defines a curve, the envelope of this line. It is the equation in the axial system (λ, θ).

$$\boxed{x = \lambda(\theta) + y.\tan(\theta)} \qquad [3]$$

The angle θ is the parameter and (x, y) are the cartesian coordinates but we can use another parameter t to define λ and θ. The coordinates of the point where this line is in contact with the envelope is given by :

$$\boxed{x = \lambda(\theta) - \lambda'_\theta \sin\theta\cos\theta \quad \text{and} \quad y = -\lambda'_\theta \cos^2\theta \quad [4]}$$

The curve in axial coordinates $\lambda = \lambda(\theta)$ is the envelope of the line [3].

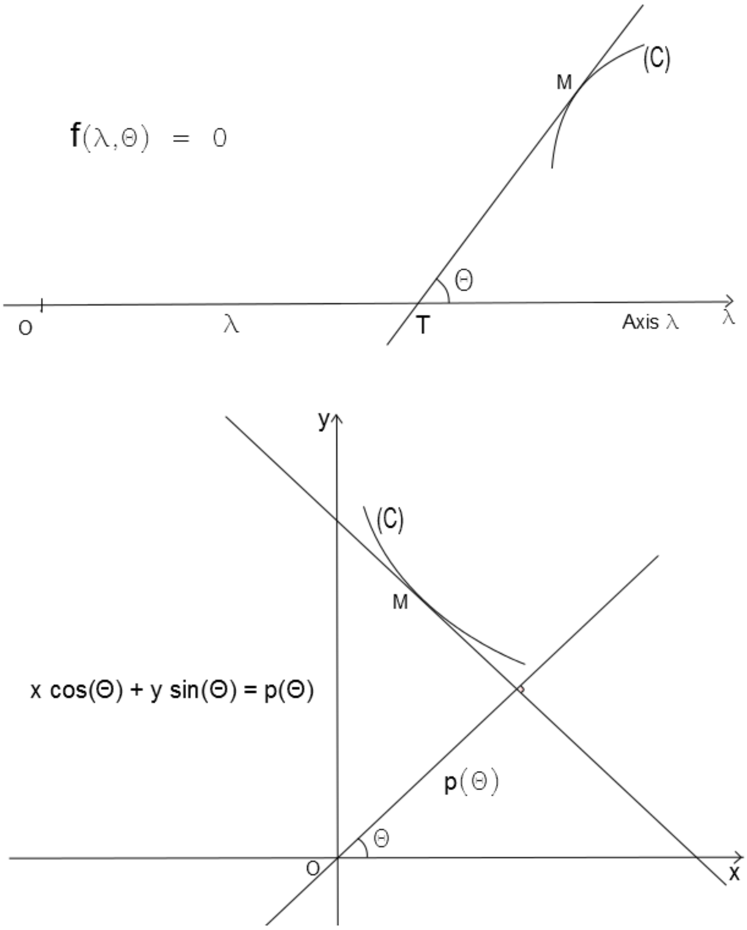

Figure 6: Axial coordinates and Euler polar tangential coordinates

4.1 Axial coordinates and TSDR :

The TSDR is easily expressed in axial coordinates. If we take the axis of TSDR as the axis (λ-axis), $\lambda(\theta)$ first coordinate and the angle θ, the second coordinate. The first coordinate λ is conserved in the TSDR so only θ is modified by the formula:

$$\tan(\theta/2).\tan(\theta'/2) = k$$

So we dispose of an equation of the curve in the form $\lambda = f(\theta)$ then the preceeding relation gives the new value of the parameter θ'.

Remark : the relation between corresponding angles in the TSDR (product of half angles tangents) is not easy to use in computations. That can explain the limited success of this transformation. But in axial coordinates if we know the equation $(\lambda(t), \theta(t))$ the curve transformed by TSDR is $(\lambda(t), \theta'(t))$ with θ and θ' linked by the above equation.

4.2 Some curves in axial coordinates.

M. d'Ocagne gave some examples of curves defined in axial system :
A point (a, b) : $\lambda = a - b.\cot\theta$
The circle (a, b, R) : $(\lambda - a).\sin\theta + b.\cos\theta - R = 0$
General conics : $(a\lambda^2 + 2b\lambda)\tan^2\theta + (2d\lambda + 2e)\tan\theta + f = 0)$
The cycloid : $\lambda = a.\theta$
The special involute of the astroid : $\lambda = a.\cos\theta$
The deltoid or Steiner tricuspid : $\lambda = a.\cos^2\theta$

4.3 An example of the analogy Inversion - TSDR :

E. Cesaro in a paper of 1885, Remarks on the paper of M. d'Ocagne, gives an example of the correlative properties of the logarithmic spiral which cuts vector rays from origin at the constant angle V. The corresponding curve with correlative properties for the TSDR is the tractrix (along a line - the asymptote), which has a constant tangent between current point (tangency to the osculator cycle) and the point (cycle of null radius) where the tangent to the current point cuts the line-asymptote.
Its axial equation (λ, θ) with $\lambda = 0$ when $\theta = \pi/2$ is :

$$\boxed{\exp(\lambda/a) = \tan(\theta/2)}$$

This is the axial correlative of equiangular spiral $\rho = \exp(a.\theta)$. In cartesian coordinates the parametric equations of the tractrix are :

$$\boxed{y = 1/\cosh\alpha \quad x = \alpha - \tanh\alpha}$$

$$\boxed{\theta\text{=Gudermanian}(\alpha) \qquad \sinh\alpha = \tan\theta \qquad \tanh(\alpha/2) = \tan(\theta/2)}$$

Cesaro gives the proposition : "The axial reciproqual of tractrix wrt its directrix (asymptote) is an equal tractrix with same directrix". It is the correlative of "the transformed by inversion of an equiangular spirals wrt to the pole (asymptotic point of the spiral) is an equal spiral".

5 Euler polar tangential coordinates and Axial coordinates are equivalent systems.

The two systems EPTC and AC define a plane curves as envelopes of a line. If we fix y-axis as the axis for AC then the angle θ can be used as the parameter of the curve (= angle with oriented Δ axis) then the equation of this line :

$$x = -y.\tan(\theta) + \lambda(\theta)/\cos\theta \qquad [5]$$

So the change is just to replace $p(\theta)$ in EPTC by $\lambda = p(\theta)/\cos(\theta)$ in AC and equations are similar.

5.1 A classical example : cycloidals.

Since a long time the circle and curves generated by circles rolling on circles have been studied. Epi- and Hypo-cycloids have many special properties when the radius of these circles have rational proportions n=p/q (with p and $q \in Z$). The EPTC equation is :

$$x.\cos\theta + y.\sin\theta - \sin(n.\theta) = 0 \qquad [6]$$

The AC equation is :

$$x = -y.\tan\theta + \sin(n.\theta)/\cos\theta$$

EPCT is the anti-pedal of the curve $p(\theta)$ and the AC is a the envelope of the line (λ, θ) depending of the the function $\lambda(\theta)$ as the variable line to define the resulting curve (C).
The curves are the cycloids generated by a point fixed on a circle rolling without sliding on another fixed circle in the plane. EPTC or axial coordinates give easy informations on parallele curves since the function $p(\theta)$ has just to be completed by a constant length $p(\theta) = p(\theta) + a$ or $\lambda(\theta) = \lambda(\theta) + a$ (a is a given length) to give all parallele curves of one of those.

6 Other examples of axial transformations.

In his book "Coordonnees axiales et paralleles" (1885) M. D'Ocagne studies the general tangential axial transformation defined by any relation between angles α and α' measured wrt the Δ-axis :

$$f(\alpha, \alpha') = 0$$

If we suppose that to each α corresponds a unique value for α' this defines an axial transformation.

6.1 Cesaro's correlative of central conchoids :

Cesaro studied the correlative of conchoid which is a central transformation known since the greeks (Nicomedes -280-210BC). From the center O the transformed point of M is the point defined by OM'= OM + a (a is a given length). He defines the axial correlative by an axis and the transformed of a semi-line is the semi-line rotated by ω a given constant angle. The transformed of a curve is the corresponding envelope.

6.2 The transformation by orthogonal tangents.

This is a special case of the preceeding case when $\omega = \pi/2$. This transformation preserves the foci. Cesaro and d'Ocagne have given some properties of this special transformation which can be expressed by the relation between angles : $\tan\theta.\tan\theta'$=-1. So $\theta' = \theta \pm \pi/2$. It is an involution $T^2 = Id$. It has links with the theory of caustics for paralleles light rays coming from infinity.
M. d'Ocagne (9) gives two theorems on orthoptics of couples of parabolas :
1 - The locus of the summit of a right angle with sides tangent respectively to 2 parabolas with common focus and same axis is the line passing through the intersection points of the parabolas.
2 - The locus of the summit of a right angle with sides tangent respectively to 2 parabolas with common focus and and orthogonal axis is the common tangent to the parabolas.
The two parabolas can be transformed one into the other by orthogonal tangents. In the first case the axis of the transformation is the line of intersection and in the second case the common tangent.

7 Conclusion.

It would be interesting to find in the plane new systems of axial coordinates, specially correlative analogs of classical central transformations like rotations, homothetie or similarity but these have to be explored in a dual space where lines or semi-lines and envelopes take place of center, points and loci of points. But analogy is not unique, leaves a large space of interpretation. So the correlative of conchoids could be called the analog of homothety since addition of angles is associated to products of exponentials. We must notice that the TSDR, that Laguerre imagined to use beside the inversion, and axial transformations have been thrown in the well of forgotten ideas waiting for a revival.

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This article is the XI^{th} part on a total of 19 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
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Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.
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Part XII : Caustics by reflection, curves of direction, Rational arc length.

Part XI : Inversion, Laguerre T.S.D.R., Euler polar tangential equation and d'Ocagne axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.
Part XIII : Catacaustics, caustics, curves of direction and orthogonal tangent transformation.
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Part XV : Rational expressions of arc length of plane curves by tangent of multiple arc and curves of direction.
Part XVI : Logarithmic spiral, aberrancy of plane curves and conics.
Part XVII : Cesaro's curves - A generalization of cycloidals.
Part XVIII : Deltoid - Cardioid, Astroid - Nephroid, orthocycloidals
Part XIX : Tangential generation, curves as envelopes of lines or circles. Arcuides, causticoides.
Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983).
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr