# TRANSLATIONS
# ROTATIONS
# ORTHOGONAL TRAJECTORIES
# GREGORY'S TRANSFORMATION
# Part - VIII

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**Abstract**

With help of Gregory's Transformation and two plane isometries : translations and rotations we examine the orthogonal trajectories of curves with parameters $x_0$ and $\theta_0$ and give some properties of families of curves associated by the GT. By analogy with the families of sinusoidal spirals and Ribaucour curves we present families of curves depending of Mc Laurin index n.

## 1 Orthogonal trajectories of Ribaucour curves for translation along x'x

The usual definition of Ribaucour curves is : cuves (C) in the plane such that if the normal of (C) M cuts a fixed line x'x at I and C is the center of cuvature of (C) at M the ratio $\frac{\overline{MC}}{\overline{MI}} = k$, the index of the Ribaucour curve. We see from this definition that x'x the base line is an axis of symmetry for the curves and we must count symmetrical curves as solutions as well. The differential equation of Ribaucour curves (Gomez Teixeira T. II) is ($n \in Z$) :

$$1 + y_x'^2 - n.y.y'' = 0 \qquad dx = \frac{dy}{\sqrt{(y)^{2/n} - 1}}$$

The sinusoidal spirals are wheels for the Ribaucour curves w.r.t. to the base-line. This property gave the definition of generalized Ribaucour curves as grounds for the curves $C_k(n,p)$ which are a generalization of sinusoidal spirals.

Orthogonal trajectories of Ribaucour curves translated along x'x are the evolutes of these curves. This can be proved by a simple geometric argument using the standard definition of these curves : if we name M the current point of this Ribaucour curve, C the center of curvature on the evolute and I the point of intersection of the normal in M with the base line x'x in the plane (fig.1).

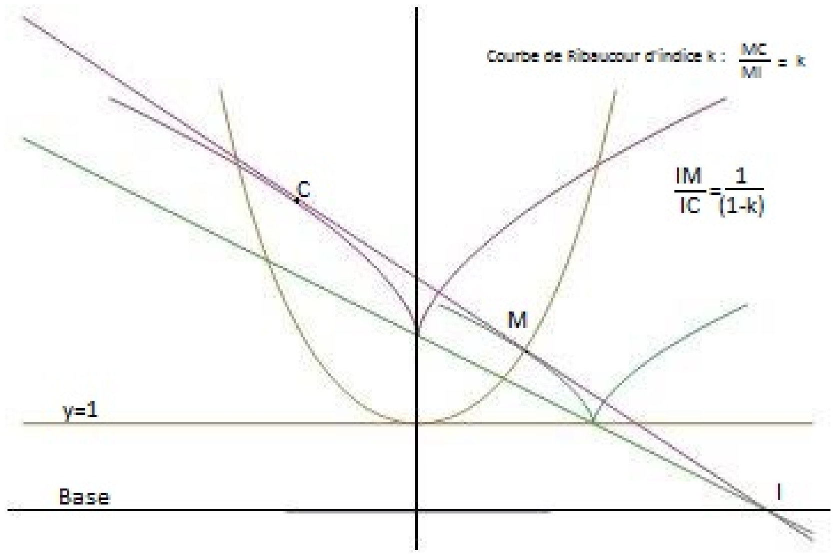


Figure 1: Orthogonal trajectories of Ribaucour curves translated along x'x are evolutes of R. C.

The above definition of Ribaucour curves is $\frac{\overline{MC}}{\overline{MI}} = k$ is equivalent to $\frac{\overline{IM}}{\overline{IC}} = \frac{1}{1-k}$ with $k \neq 1$ . A dilation centered in I with ratio $\frac{1}{1-k}$ moves C to M and transforms the evolute in a curve tangent in M to the line IC so cuts orthogonally the Ribaucour curve at M since IC is the normal. The cusp of the evolute is moved to a point on the line parallele to the base line tangent to the minimum or maximum of the curve. When the point I moves along the base line the transformed evolutes are all equals, cut orthogonally the Ribaucour curve at M and are moved by a translation parallele to the base line x'x.

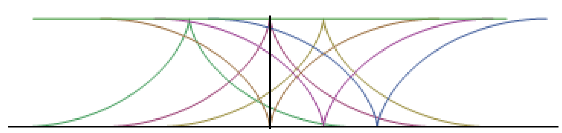

Figure 2: Cycloid and cycloid cusps upward are orthogonal curves for translation along x'x

k= 2 is the cycloid cusps on x'x, orthogonal trajectories corresponds to equal symmetric cycloids (cusps upward) since the I-homothety ratio is $1/(1-k) = -1$ (fig.2).
k= -1 is the catenary, orthogonal trajectories are I-homothetic of the evolute of the catenary (ratio:1/2).
k= -2 is the parabola. The orthogonal trajectories are I-homothetic (ratio : 1/3) of the evolute of the parabola so semi-cubic parabolas $x - x_0 = (2/3).(y-1)^{3/2}$ (fig.3).

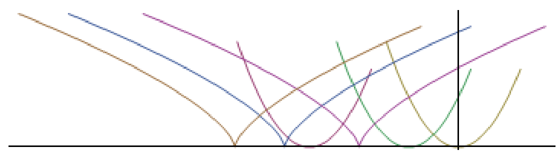

Figure 3: Parabolas and Semi-cubic parabolas are orthogonal curves for translation along x'x

A direct study of the specific case k=1 (the circle) shows that for the orthogonal curves of circles of constant radius 1 centered on x'x and translated parallely to the base line is a double tractrix with x'x as common asymptote (fig.4).

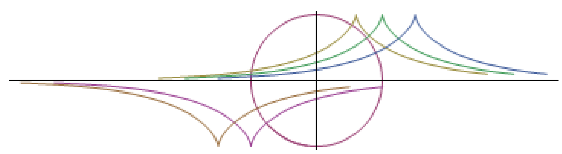

Figure 4: Circles and Tractrix are orthogonal curves for translation along x'x

We will show that these curves (one cusp spirals) :

$$\rho = (1-n)\cos^n u \quad \theta = n.[\tan u - u] \quad (n \neq 1)$$

are wheels for the evolutes of Ribaucour curves with respect to the base line and these one cusp spirals and the sinusoidal spirals are two families of orthogonal curves turning around the pole O. Note that the involute of the circle and the tractrix spiral are element of the class of one cusp spirals.

## 2 Parabolas/Hyperbolas $\frac{n+1}{n}x = y^{\frac{n+1}{n}}$ and $\theta/n = \rho^{1/n}$ spirals

General Parabolic or Hyperbolic spirals belong to the general class of curves $C_1(n,p)$, which depends on two rational parameters : n and p (see Part III for details) :

$$\boxed{\rho = \frac{\sin^{(p+n)}(u)}{\cos^n(u)} \qquad \theta = n\tan(u) + p.u}$$

If we take pedal index p=0 and Mc Laurin index n=n then :

$$\rho = \frac{\sin^n(u)}{\cos^n(u)} \qquad \theta = n\tan(u) + 0.u$$

$$\rho = \tan^n u \qquad \theta = n.\tan u$$

or :

$$\rho^{1/n} = \tan u \qquad \theta/n = \tan u$$

or :

$$\rho = (\theta/n)^n$$

These parametric equations are normalized in such a way that $\tan V = \tan u$. This is the subclass $C_1(n, 0)$ and from now we call them parabolic/hyperbolic spirals.
In the same way we normalize the parabolas/hyperbolas in a plane xy. These are grounds for the parabolic/hyperbolic spirals as wheels :

$$x - x_0 = \int \rho.d\theta = \frac{n}{n+1}.y^{\frac{n+1}{n}} \qquad \text{and so} \qquad \tan V = \frac{dx}{dy} = \tan u = t$$

or in a more symmetrical form :

$$x = \frac{n}{n+1}.t^{n+1}, \quad y = t^n$$

## 2.1 Orthogonal trajectories of Parabolas/Hyperbolas

We search for the orthogonal trajectories of the parabolas/hyperbolas of equation $\frac{n+1}{n}(x - x_0) = y^{\frac{n+1}{n}}$ ($n \neq 0$, -1 not necessary integer) translated along x'x by the distance $x_0$. The equation of orthogonal curves obtained by the usual method ($dy/dx \to -dx/dy$) is :

$$(x - x_0) = \frac{n}{(1-n)}.y^{\frac{n-1}{n}} \qquad n \neq 1, 0$$

So the orthogonal trajectories of parabolas/hyperbolas are curves of the same family.
n=2 is the semi-cubic parabola (the evolute of the parabola) : $x - x_0 = (2/3).y^{3/2}$ which has for orthogonal trajectories the parabolas $x - x_0 = -2.y^{1/2}$ with vertical axis.

n= 1 is a special case and corresponds to the parabolas $2p.(x - x_0) = y^2$ with $p = 1$. A specific computation leads to the orthogonal curves: $y = e^{-p(x-x_0)}$, exponential curves translated along its asymptote the x'x axis (see fig.5). The two orthogonal curves have equal constant p subtangent (exponential) and subnormal (parabola).

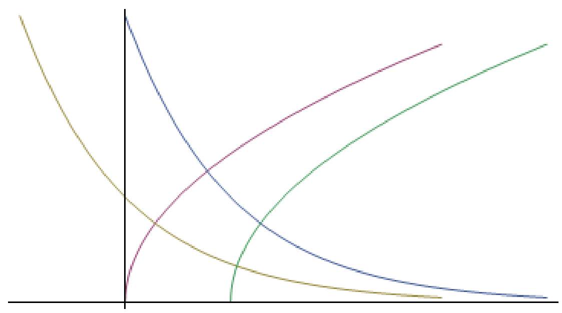

Figure 5: Exponentials $e^{-p.x}$ and Parabolas $y^2 = 2.p.x$ translated along x'x are orthogonal curves

## 2.2 Orthogonal trajectories of Parabolic/Hyperbolic spirals

We search now for the orthogonal trajectories of $n.(\theta - \theta_0) = \rho^n$ spirals for rotation around the pole O by angle $\theta_0$ in polar coordinates. The equation of orthogonal curves obtained by the usual method $[\frac{d\rho}{d\theta} \rightarrow -\frac{\rho^2.d\theta}{d\rho}]$ is :

$$n.(\theta - \theta_0) = \rho^{-n} \qquad n \neq 0$$

So the orthogonal trajectories of parabolic/hyperbolic spirals are curves of the same family. If n=0 we obtain $\theta - \theta_0 = C$ (constant) so it gives lines passing through O and orthogonal curves are circles with center at O.
n=1 corresponds to the spiral of Archimede $\theta - \theta_0 = \rho$ the orthogonal curves are Hyperbolic spirals $\theta - \theta_0 = \rho^{-1}$. At a point of intersection two orthogonal curves have equal polar-subnormal (Spiral of Archimede) and polar-subtangent (Hyperbolic spiral).

## 2.3 Parbolic/Hyperbolic spiral wheels and parabolas/hyperbolas grounds

As we have seen above using direct Gregory's transformation ($x = \int \rho.d\theta, y = \rho$) the curves $n.\theta = \rho^n$ are wheels for grounds with equations :

$$x - x_0 = \frac{n}{(n+1)}.y^{\frac{n+1}{n}}$$

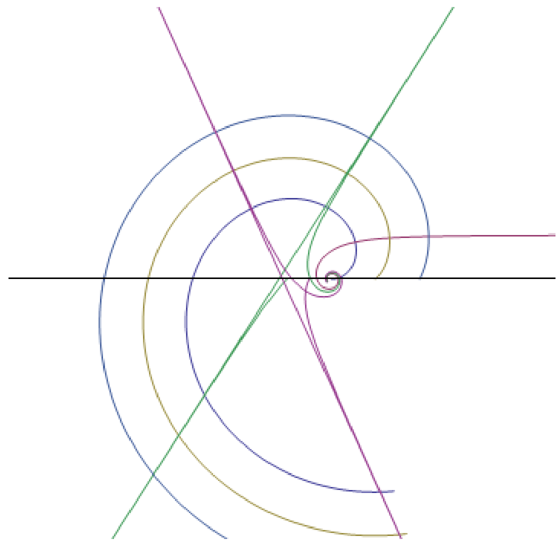

Figure 6: Archimede and hyperbolic spirals are orthogonal curves for rotation around the pole

The grounds for parabolic or hyperbolic spiral wheels are parabolas and hyperbolas. A classic example is the spiral of Archimede $\theta = \rho$ wheel for the parabola with axis as the base line. Gregory's transformation and orthogonal trajectories give the same families of curves.
For n=-1 the wheel is an hyperbolic spiral $\theta = 1/\rho$. A specific computation leads to an exponential ground $y = e^{(x-x_0)}$, the asymtote is the base line x'x.
Al these examples illustrate the general dual properties at the heart of Gregory's transformation (cf part I) for curves in polar coordinate with pole O and curves in orthonormal coordinates with base line x'x. Rotations around O and translations along the base line x'x are dual elements.

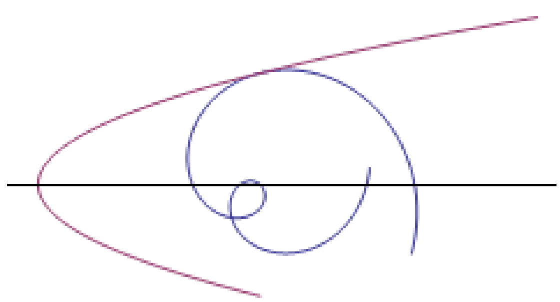

Figure 7: Archimede spiral $\rho = p.\theta$ as wheel for the parbola ground : $y^2 = 2.p.x$.

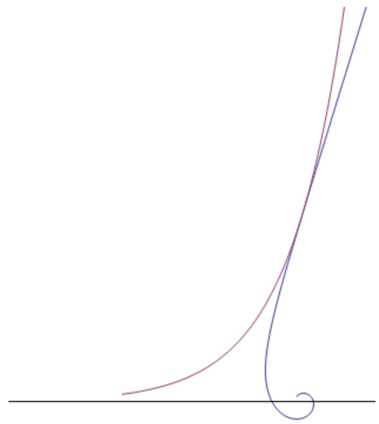

Figure 8: Hyperbolic spiral as wheel for the exponential ground (base line : asymptote).

# 3 Families of curves defined by Gregory's tranformation and orthogonal trajectories

If $y(t)$ is a smooth function we associate two curves in the plane one in orthonormal coordinates $(x, y)$ and the second in polar coordinates $(\theta, \rho)$ in the following way with Y(t) a primitive of y(t) :

$$(x, y) = (\int y(t).dt, y(t)) = (Y(t), y(t)) \quad \text{and} \quad (\theta, \rho) = (t, \rho(t) = y(t))$$

These equalities define a couple of associated ground and wheel associated by Gregory's transformation (direct GT). Or in the other way ($GT^{-1}$) we need two functions $y(t) \neq 0$ and $x(t)$ :

$$(x, y) = (x(t), y(t)) \quad \text{and} \quad (\theta, \rho) = (\int \frac{x'(t).dt}{y(t)}, y(t))$$

We search orthogonal trajectories of curves moved by translation or rotation, one for the ground-curves $[x(t) - x_0, y(t)]$ for translations along x'x axis and the second for the corresponding wheel-curves $[\theta(t) - \theta_0, \rho(t)]$ for rotations around the pole O.

## 3.1 Families of curves defined by translations, rotations and differential equations.

From an initial family of curves we can associate by Gregory's Transformation and orthogonal trajectories three other families with x, y, $\theta$, $\rho$ and a parameter

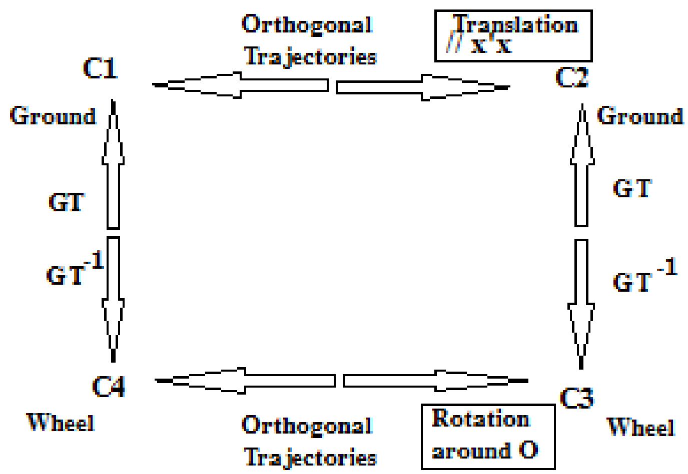


Figure 9: Orthogonal trajectories - Grounds and wheels

u proportional to $V(= k.u)$.

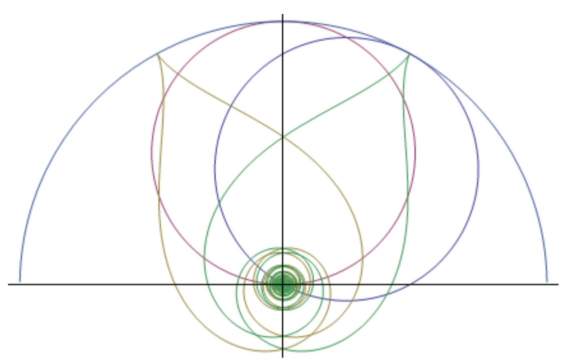

Figure 10: Sinusoidal spirals (circle) and one cusp spirals (spiral tractrix) are orthogonal trajectories : n=1

The differential equations are for :
- Ribaucour curves with is :

$$\frac{dx}{dy} = \frac{1}{\sqrt{y^{2/n} - 1}} = f(y) = \tan V$$

- Sinusoidal spirals is :

$$\frac{\rho . d\theta}{d\rho} = \frac{1}{\sqrt{\rho^{2/n} - 1}} = f(\rho) = \tan V$$

## 3.2 Inversion of the wheel - Transformation "T"

For grounds and wheels curves we set $y = \rho = \cos^n u$ where the integer n is Mc Laurin index. If we take the opposit index -n it is an inversion for the wheel and so $\rho \to 1/\rho$ and $\theta \to \theta$. The first differential equation of sinusoidal spirals becomes with $\rho \to 1/\rho$ and $d\rho \to -d\rho/\rho^2$ :

$$\frac{\rho . d\theta}{d\rho} = \frac{-\rho^{2/n}}{\sqrt{1 - \rho^{2/n}}} = -\tan V$$

With help of Gregory's transformation (GT: $x = \int \rho d\theta$ and $y = \rho$) we define a new one "T" $= GT\circ$ (Inversion/O) $\circ GT^{-1}$ on the ground $C_1$ with its base line x'x in the following way. First we take the associated wheel $C_2$ by $GT^{-1}$ then the inverted of this wheel in the inversion of pole O : $C_3$. And to finish we take ground associated to $C_3$ by GT this gives ground $C_4$. This is the transmuted of an inversion of pole O by Gregory's Transformation. We could assimilate this transformation to an inversion with base line instead of pole O. This transformation is an involution $T^2 = Id$.

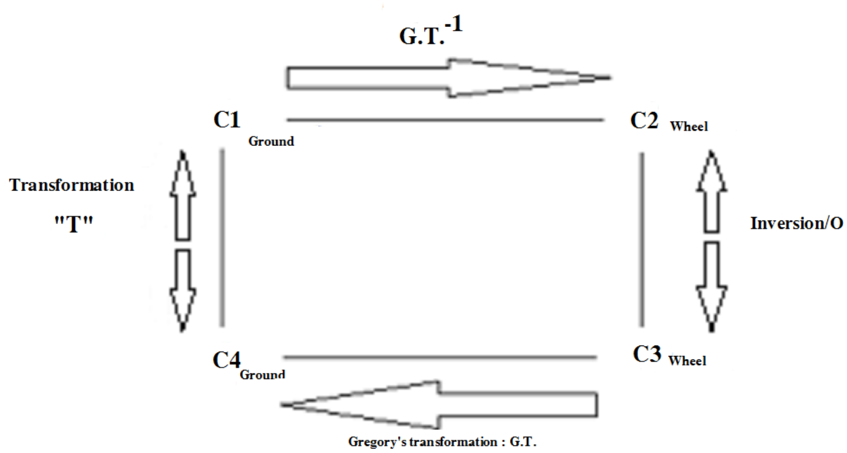


Figure 11: "T" : transformation from ground to ground

An example will illustrate this definition of "T". If $C_1$ is a circle of Cardan centered on x'x ($x = \sin\theta$, $y = \cos\theta$), $C_2$ is the circle $\rho = \cos\theta$, the inverted of

this circle is the line $C_3 : \rho = 1/\cos\theta$. The associated ground $C_4$ is the catenary $y = \cosh x$.
So the circle and the catenary are associated by "T".
The differential equation of Ribaucour curves and sinusoidal spirals indicated by Gomez Teixeira are :

$$\frac{dx}{dy} = \frac{1}{\sqrt{y^{2/n} - 1}} = \tan V$$

$$\frac{\rho.d\theta}{d\rho} = \frac{1}{\sqrt{\rho^{2/n} - 1}} = \tan V$$

with the Mc Laurin index n. But transformation "T" associates, for same n, curves of the same family with $y \leq 1 = y_{max}$ to a curve with $y \geq 1 = y_{min}$. We just change y $\rightarrow \frac{1}{y}$ and $dy \rightarrow -\frac{dy}{y^2}$ so the new equation is :

$$\frac{dx}{dy} = \frac{-y^{-2+1/n}}{\sqrt{1 - y^{2/n}}} = -\tan V$$

To replace n by -n of the Ribaucour curve is equivalent to apply transformation T above and a Ribaucour curve with $y \leq y_{max}$ becomes a Ribaucour curve with $y \geq y_{min}$. A circle centered on base-line x'x becomes a Catenary, a cycloid a parabola, and line y=constant another line parallele to x'x, etc...
A peculiar solution of the differential equation of Ribaucour curves depend on initial conditions ($x_0$, $y_0$, $V_0$) and parameter n :
1- If $n < 0$ then $|y(t)| \geq y_{min}$ the curves are above the $y = y_{min}$ or under $y = -y_{min}$ since the definition of Ribaucour curves has the base line x'x for symmetry.
2- If $n \geq 0$ then $|y(t)| \leq y_{max}$ the curves are in the strip between $y = -y_{max}$ and $y = y_{max}$.
3- There are also the singular integrals of the differential equation : the lines $y = \pm y_{max}$ or $y = \pm y_{min}$ that are enveloppes of the Ribaucour curves. And we must consider these two lines as special solutions.

### 3.3 Orthogonal trajectories

For the orthogonal trajectories we need usual formulas :

$$\frac{dx}{dy} \longrightarrow -\frac{dy}{dx} \qquad \frac{\rho.d\theta}{d\rho} \longrightarrow -\frac{d\rho}{\rho.d\theta}$$

The latter is equivalent to $\frac{d\rho}{d\theta} \longrightarrow -\frac{\rho^2.d\theta}{d\rho}$
If we apply this to the differential equation of Ribaucour curves and to the differential equations of the sinusoidal spiral the corresponding differential equations (with separated variables) for orthogonal trajectories are respectively :

$$dx = -\sqrt{y^{2/n} - 1}.dy$$

$$\rho.d\theta = -\sqrt{\rho^{2/n}-1}.d\rho$$

The first one gives the evolutes of Ribaucour curves as we have seen above they are the orthogonal trajectories of the Ribaucour curves translated along x'x. The second equation has for solutions the one cusp spirals of class $C_1(n, -n)$ in polar parametric coordinates :

$$\rho = \cos^n u \qquad \theta = n.[\tan u - u] + \theta_0$$

and these spirals are orthogonal to sinusoidal spirals turning around the pole O.

$$\frac{d\rho}{d\theta} = \frac{-\cos^{n+1} u}{\sin u} \quad \text{and} \quad \rho = \cos^n u$$

$$\frac{d\rho}{d\theta} = \frac{-\rho^{1+1/n}}{\sqrt{1-\rho^{2/n}}}$$

(n = Mc Laurin index) then :

$$d\theta = -\frac{\sqrt{1-\rho^{2/n}}}{\rho^{1+1/n}}.d\rho = -(1/\rho).\sqrt{\rho^{2/n}-1}.d\rho$$

This confirms the fact proved by a geomtric argument in the first section : orthogonal trajectories of sinusoidal spirals turning around the pole are one cusp spirals. These cusp spirals are wheels for the evolutes of Ribaucour curves w.r.t. to the base.

## 3.4 Polarity w.r.t. a circle

Transformation "T" is formally similar to the transformation by reciprocal polar P with respect to a circle of center O which can be defined by the product $(pedal/O) \circ (inversion/O, 1) \circ (pedal/O)$. This polarity is also an involution $P^2 = Id$ and is conformal so the angle V is preserved for transformed curves (mod a symmetry). For the curves $C_k(n, p)$ (see part III) a polarity corresponds to the changes $n \to -n$ and $p \to -p-1$ [pedal : $(n, p+1)$ and inversion: $(-n, -p-1)$].
An example in class $C_k(n, p)$ : initial curve is the involute of the circle $C_1(-1, 0)$ $\rho = 1/\cos u$, $\theta = \tan u - u$, pedal is $C_1(-1, 1)$ $\rho = \tan u$, $\theta = \tan u = \rho$ the spiral of Archimede, the inverse is the Hyperbolic spiral $C_1(1, -1)$ $\rho = 1/\tan u$, $\theta = \tan u = \rho$ and the pedal is the tractrix spiral $C_1(1, 0)$ $\rho = \cos u$, $\theta = \tan u - u$.

## 3.5 General form of differential equations for translations or rotations

We list a few cases that can be found in the present paper or among special types of generalized curves (cf Part III and VII). They correspond to cases where the expression of $\pm dx/dy = \tan ku$ or $= 1/\tan ku$. The parametrization by the parameter u proportionnal to angle V, preserved in Gregory'transformation, gives simple polar equations for the curves $C_k(n,p)$.

We give the name of the family and corresponding differential equation in (x,y) or $(\theta,\rho)$ which are in the form (n= Mc Laurin index):

$$dx = f_n(y).dy \quad \text{for the grounds and} \quad d\theta = \frac{f_n(\rho)}{\rho}.d\rho \quad \text{for the wheels and} \quad f_n(t) = \tan V$$

We give also natural for a parametrization with trigonometric or hyperbolic functions : u a circular angle and $\alpha$ a hyperbolic argument $[u = Gd(\alpha)]$.

1- [Ribaucour curves - Sinusoidal spirals] :

$$f_n(t) = \frac{1}{\sqrt{t^{2/n}-1}} = 1/\tan u = 1/\sinh\alpha \qquad t = \cos^{-n} u \qquad t = \cosh^n \alpha$$

2- [evolutes of Ribaucour curves - One cusp spirals]

$$f_n(t) = -\sqrt{t^{2/n}-1} = -\tan u = -\sinh\alpha \qquad t = \cos^{-n} u \qquad t = \cosh^n \alpha$$

3- [Pursuit curves - Anallagmatic spirals] :

$$f_n(t) = \frac{t^{2/n}-1}{2.t^{1/n}} = \frac{[t^{1/n} - t^{-1/n}]}{2} = \sinh\alpha \qquad t = e^{n.\alpha}$$

4- [$\beta-$curves - Tangentoid spirals ]:

$$f_n(t) = \frac{2.t^{1/n}}{1-t^{2/n}} = \frac{-2}{[t^{1/n} - t^{-1/n}]} = -1/\sinh\alpha \qquad t = e^{n.\alpha}$$

5- [Parabolas/Hyperbolas - Parabolic/Hyperbolic spirals ]

$$f_n(t) = t^{1/n} \qquad t = \tan u$$

In this last class of curves orthogonal trajectories for translations along x'x axis and rotations around pole O are curves of the same class.

# 4 References

There are 8 papers on Gregory's transformation and related topics.
Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part II : A generalisation of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta-$curves.

Part VIII : Translations, rotations, orthogonal trjectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, closed or periodic curves and Gregory's transformation.

There are two papers in french :
- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (No 15 - Fevrier 1983)
- Une generalisation de la roue - Bulletin de l'APMEP (No 364 juin1988).
All are on my web page : http://christophe.masurel.free.fr

Books :
- Courbes geometriques remarquables 'H. Brocard , T. Lemoine) Blanchard Paris 1967 (3 tomes)
- Traite des courbes speciales remarquables (F. Gomez Teixeira) Chelsea New York 1971 (3 tomes)
- Nouvelles annales de mathematiques (1842-1927) Archives Gallica
- Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
- Geometriae pars universalis (James Gregory) Padova 1668.