TSCHIRNHAUSEN'S CUBIC
- Part IV -

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Abstract

We present some properties of the Tschirnhausen's Cubic related to the couples ground-wheel and gregory's transformation.

1 Definition :

Tschirnhausen's cubic (TC) has been presented in 1682 in Acta Eruditorium by Walter Tschirnhausen as the caustic by reflection of the parabola for light rays perpendicular to the axis of symmetry. Another definition of (TC) is the antipedal of the parabola w.r.t. its focus.
It is also the third pedal of the point in polar coordinates :
- Line : $\rho = \frac{1}{\cos(\theta)} \leftarrow$ first anti-pedal of a point.
- Parabola : $\rho = \frac{1}{\cos^2(\theta/2)} \leftarrow$ second anti-pedal
- Tschirnhausen's cubic: $\rho = \frac{1}{\cos^3(\theta/3)} \leftarrow$ third anti-pedal
These are three sinusoidal spirals $\rho = \cos^n(\theta/n)$ with n $\in$ Z)
Parametric equations (x, y)-plane of this cubic are :

$$x = t - t^3/3 \qquad y = t^2$$

2 Cartesian and polar equations of the TC

We need the equation of the TC with vertical axis in an orthonormal frame in the following form :

$$x = t - t^3/3 \qquad y = 1 + t^2$$

and :

$$\tan(V) = \frac{dx}{dy} = \frac{1 - t^2}{2t}$$

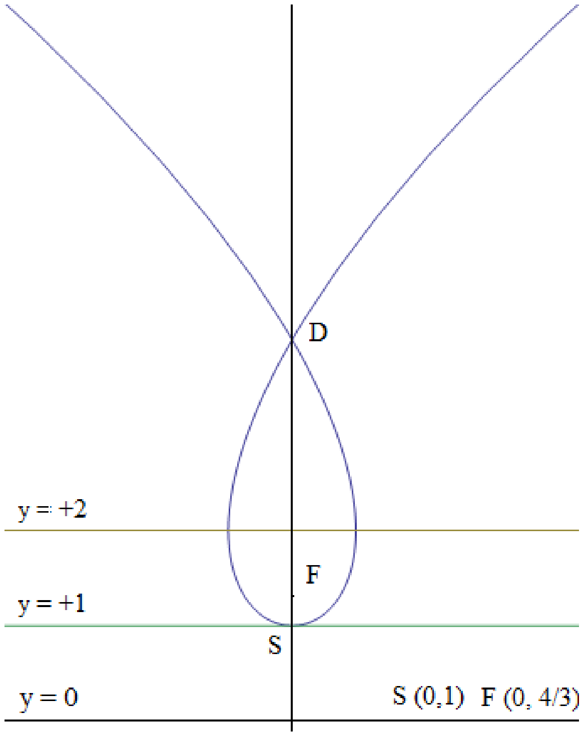

Figure 1: Tschirnhausen's Cubic

The coordinates of the focus F is (0, 4/3), the summit S(0, 1) and the double point D(0, 4). The parameter $t = \tan u$ implies $V = \pi/2 - 2u$. The polar equation with the pole at the focus of the TC is:

$$\rho = \frac{1}{3\cos^3(\theta/3)} \qquad \text{with} \quad t = \tan(\theta/3)$$

Three horizontal lines in the plane of TC play a particular role in the geometry of the curve:
The directrix/natural base-line (y=0), the tangent at S (y=+1) and the double normal (y=+2).

3 Equation w.r.t. the focus and the directrix

The TC(ρ,y) can be expressed in the frame : distance ρ to the focus and y to the directrix the same used for conics. With the choice we have made in the equation of the section 2. The directrix is the axis x'x. We have seen above that $u = \theta/3$

$$1 + t^2 = \frac{1}{\cos^2(u)} = y$$

$$\rho = \frac{1}{3\cos^3(\theta/3)}$$

So the equation focus-directrix is : $9\rho^2 = y^3$

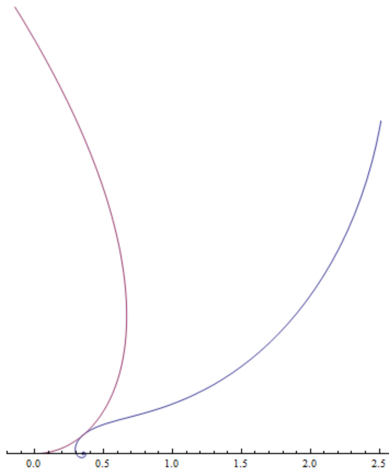

Figure 2: Curve $[\rho = t^2 \quad \theta = t + 1/t]$ rolling on a half-TC base line is tangent at S

4 The TC as a pursuit curve

The pursuit curves for a mobile running along a line are caustics by reflection of parabolas/hyperbolas $y = x^k$ for light rays parallel to x'x axis. For the general case the equation of the pursuit curve is :

$$x = \frac{1}{2}\Big[\frac{1}{m+1}t^{m+1} + \frac{1}{m-1}(t^{1-m})\Big] \qquad y = t^{1/m}$$

For the caustic by reflection with light rays parallele to xx' : m= k-1. The parametric equations abovedefine a pursuit curve for a speed of the prosecutor 1/m times the speed of the mobile on the line. For the case of the parabola, the pursuit curve corresponds to m=1/2. The speed of the pursuiter is twice the speed of the mobile that runs on the tangent line at the summit S is the TC. So the first one meets the mobile at the point S of contact of the tangent with the TC then the pursuit curve is the TC :

$$x = t - t^3/3 \text{ and } y = t^2$$

The wheel associated to this tangent as the base-line is a special case of anallagmatic spirals and is given in parametric polar coordinates by :

$$\rho = t^2 \text{ and } \theta = t + \frac{1}{t}$$

5 Properties derived from the system ground-wheel

In this section we use two base-lines y=0 (the directrix) and y=+2 (the double normal) and the inverse Gregory's transformation (GT^{-1})
We search for the wheel in parametric polar coordinates and for the first one (y=0) we find :

$$\rho = y = 1 + t^2 = \frac{1}{\cos^2(u)} \quad \text{and} \quad \theta = \tan(u) - 2u$$

This curve is the Sturm or Norwich spiral, an involute of the involute of the circle. For the other base-line (y=+2) and the same TC the wheel is :

$$\rho = t^2 - 1 \quad \text{and} \quad \theta = t \quad \text{so} \quad \rho = \theta^2 - 1$$

These two wheels can roll one on the other around two poles at the distance of 2 with usual conditions. When the Norwich spiral rolls along the xx' axis the pole describes the curve:

$$x = t - t^3/3 \text{ and } y = t^2 - 1$$

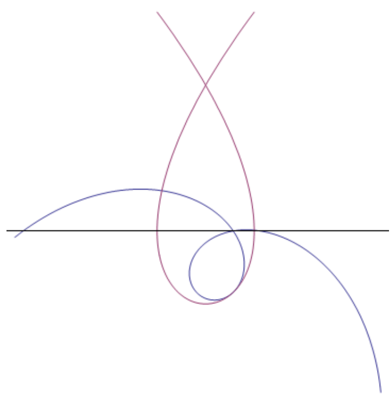

Figure 3: Curve $\rho = \theta^2 - 1$ rolling on a TC base line is double normal

It is a TC for the base-line moved of a length 2 downwards. The pedal of the Norwich Spiral is the curve $\rho = \theta^2 - 1$. So with the theorem 8 (see part (I) Gregory's transformation) : if the Tschirnhausen's Cubic rolls with its linked base-line y=0 on a line then the enveloppe of the base line is the

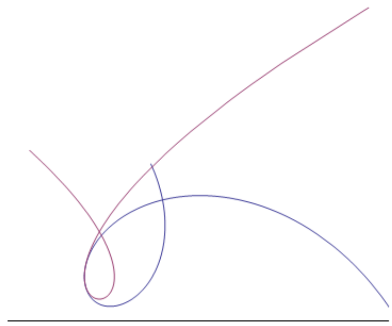

Figure 4: Sturm/Norwich spiral-wheel and TC-ground base line is the directrix

roulette of the pole of the Sturm/Norwich spiral.
It can be shown that this roulette of this spiral is Tschirnhausen's Cubic :

$$x = t - t^3/3 \qquad y = t^2 - 1$$

for which the base-line is the double normal. So we have the following result:
Theorem : If a Tschirnhausen's Cubic rolls on a line then the directrix has for envelope an equal TC.
Steiner Habich theorem leads also to this property if we note that the pedal of the Norwich Spiral is $\rho = \theta^2 - 1$ which is a wheel for the TC as the ground and double normal as the base-line.

6 TC as the caustic of the parabola

The property of the TC to be the caustic by reflection of the parabola is proven in any book about this curve for the light rays orthogonal to the axis of the parabola. Knowing the result for this particular direction of the rays it is possible to prove that the caustic is also a TC when the parallel light rays come from any direction except for the one in the direction parallel to the axis of the parabola. In this last case the reflected rays converge to the focus of the parabola. Let's suppose the parabola is : $y^2 = 2px$ and the light rays come from ∞ in the direction φ angle with the x'x axis (axis of the Parabola). There is a similarity centered at the common focus of the parabola and the TC of angle $\pi/2 - \varphi$ and ratio $\sin(\varphi)$ which transforms the parabola in the curve described by the projection H of the focus on the reflected ray. It is a parabola and the line HM has for envelope another TC anti-pedal of this last parabola transformed from the initial one by the same similarity of angle $\pi/2 - \varphi$ and ratio $sin(\varphi)$.The distance from F to the summit S of the TC is $a_\varphi = a_{\pi/2}.\sin(\varphi)$.

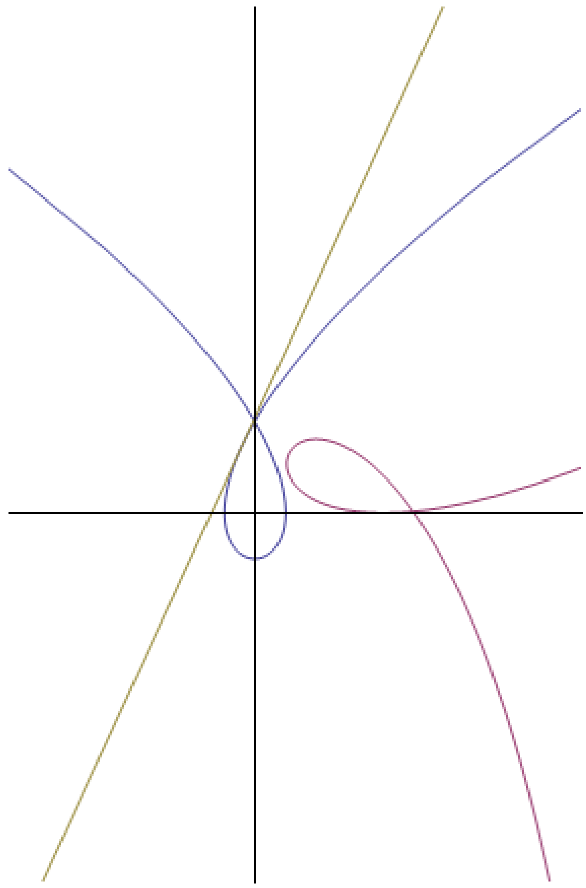

Figure 5: The envelope of linked directrix of TC rolling on a line is an equal TC

7 Curves of Duporcq - Sturmian spirals , Tschirnhausen's cubic and Norwich spiral

In four papers of NAM , E. Duporcq, A. Manheim (1902), F. Balitrand (1914) and M. Egan (1919) gave the complete solution of the following problem for plane curves : to find the curves such that the element of arc length is e times (e= constant) the segment cut on a given line by the normals at the ends of this arc. If M is the current point of the TC :

$$x_M = t - t^3/3 \qquad y_M = t^2 - 1$$

The abcisse of N on the normal of the TC at M for y=0 is :

$$x_N = t + t^3/3 = \text{ the arc length of TC between t=0 and t=t.}$$

So $s(A, B) = \Big[t + t^3/3\Big]_{to}^{t1}$ is the arc length between two points A and B on

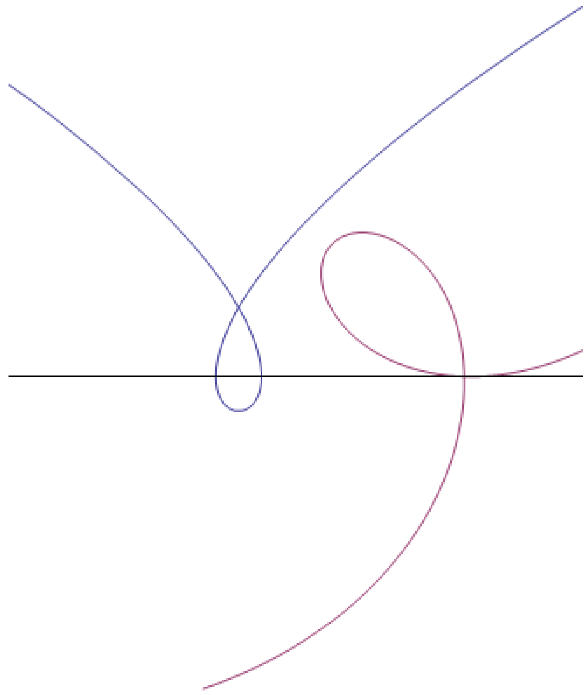

Figure 6: Roulette of Norwich (sturmian) spiral is a Tschirnhausen's cubic (Duporcq curve).

the TC. This result confirms that TC is the solution of Duporcq's problem when e=1 : equality of the arc and the segment cut by the current normal on the double normal y=0 (fig.7).

7.1 Duporcq and Bouguer

The sturmian spirals are the curves such that $R_c = e\rho$: ray of curvature equal to e times the vector ray in polar coordinates. Mannheim has shown that the roulette of the pole of a Sturmian spiral is a Duporcq curve.
The special case e=1 is the sturmian spiral of Norwich the pole of which describes a Duporcq curve : a Tschirnhausen's cubic when rolling on the double normal of the TC as base line.
The analog problem - but for the tangent - (the arc of the curve is k times, k= speed ratio, the segment cut by the tangents at the ends of the arc on a given line) for a plane curve are the Bouguer curves of pursuit (see Part VII).
And so the TC is a Bouguer pursuit curve for the tangent at the summit with speed ratio k=2 and a curve of Duporcq for the double normal for e=1 (fig. 7). The couple TC - spiral of Norwich is a special case among the

solutions given by M. Egan (NAM 1919) - see part IX -.

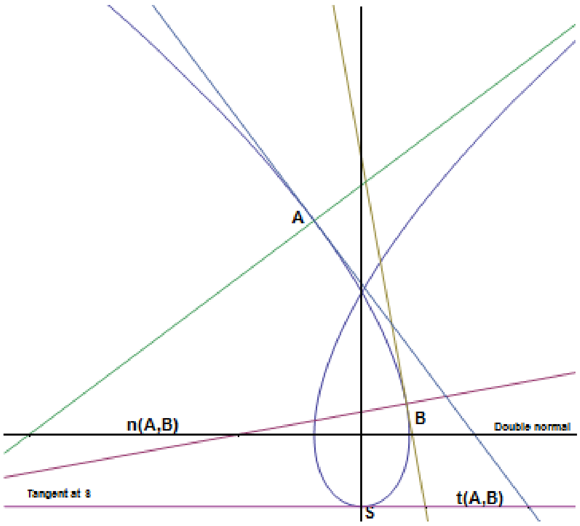

Figure 7: Tschirnhausen's cubic as Duporcq curve and as Pursuit curve. Normals and Tangents at A and B : $s(A, B) = n(A, B) = 2.t(A, B)$

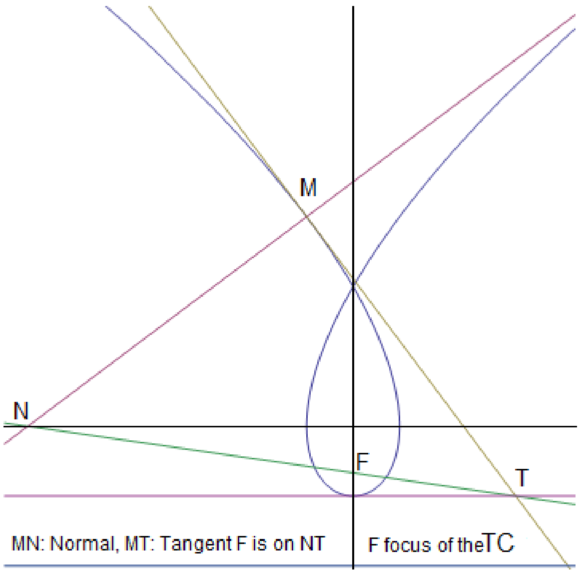

Figure 8: A property of normal and Tangent at M

7.2 A property of the line NT

The normal at M to Tschirnhausen's cubic cuts the double normal in N and the tangent at M intersects the tangent at S in T. Since $x_N = -2.x_T$ the focus $F(-2/3, 0)$ is on the line NT (fig.8).

8 A property of the osculating circle of the TC

In a paper of 1902 (NAM Sur une classe de courbes planes remarquables) Cesaro gives the intrisic equation of general curves of Cesaro (λ, μ : real parameters):

$$s = \int \frac{\lambda}{\sqrt{(\rho/a)^{\mu} - 1}} d\rho$$

Cesaro also defines a subclass of his curves by the property that the dilation of the osculator circle in a constant ratio k from the current contact point is tangent to a fixed line in the plane. This subclass corresponds to $\lambda = \mu$. The TC belongs to this class for the ratio k = - 1. So the symetric of the osculating circle at M $\in$ TC w.r.t. to the current point or the tangent at the same point M is tangent to the directrix.
Any curve of this class is the envelope of circles centered on a Ribaucour curve and tangent to the directix and, when it is algebraic, is a curve of direction. Poleni's curve (k= 2) and Nephroid (k = 2/3) are other examples of this class.

9 Tschirnhausen's Cubic as a curve of direction

Wieleitner showed in (9) that the same TC is the caustic by reflection, for parallele incident light rays, of a parabola and of a semi-cubic parabola. The focus of the three curves is is the same point F. The common catacaustic is the envelope of circles centered on the the parabola and semi-cubic parabola and tangent to x-axis. In fig 9 the four curves (parabola, Semi cubic parabola (SCP), catacaustic/hypercycle and caustic (=TC) :

$$x = -t^2 \qquad y = 2t$$

$$x = 3t^2 \qquad y = 2t^3$$

$$x = \frac{t^2(3 - t^2)}{1 + t^2} \qquad y = \frac{4t^3}{1 + t^2}$$

$$x = -3t^2 \qquad y = -3t + t^3$$

The two first curves are associated by the tangential transformation called "by orthogonal tangents" since for a couple of touching circles (see fig. 9) on the catacaustics, the tangents at the centers on parabola and SCP are orthogonal and their intersection lies on the x-axis.
This property is true for the general parabolas and their caustic by reflection. A parabola $y = ax^n$ can be associated with its transformed "by orthogonal

tangents” with the x-axis and the two circles centered at the touching points are tangent on the common catacaustic. M. d’Ocagne showed that this transformation preserves the foci of curves. The evolute of this catacaustic is a curve of pursuit of Bouguer and a curve of direction for rational values of n.

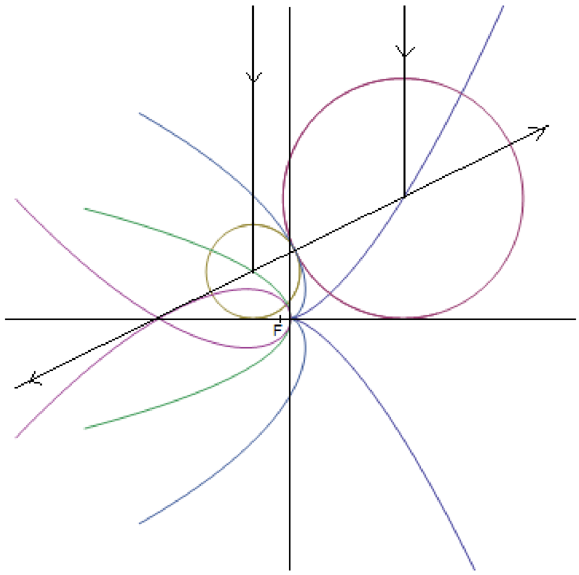

Figure 9: Tschirnhausen cubic as curve of direction for Parabola and semi-cubic parabola

The caustic of the Tschirnhausen’s cubic for light rays coming from the focus is a semi-cubic parabola.
In fig 10 the three curves : TC, parabola/catacaustic and the caustic = Semi cubic parabola (SCP) :

$$\rho = 1/\cos^3(\theta/3)$$

$$\rho = 2/\cos^2(\theta/2)$$

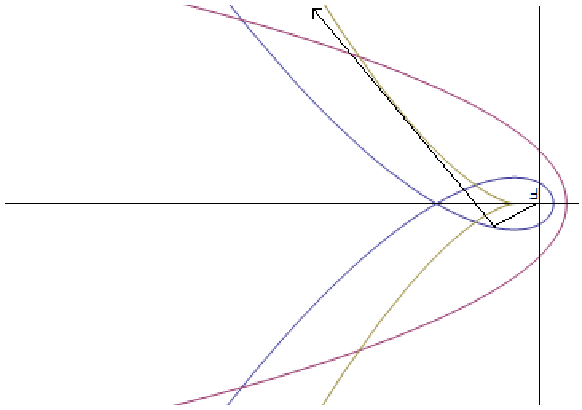

Figure 10: Caustic of a TC for light rays diverging from focus

Annexe I : Roulette, Pedal and Gregory's transformations.

Parametric equations of the roulette (x, y) of the pedal (ρ, θ) and Gregory's transformations (GT, GT^{-1})
Roulette : the formulas to determine the cartesian parametric equation (x, y) of the roulette on the line xx' of a curve given in polar coordinates (ρ, θ):

$$y = \rho \sin(\theta) \text{ and } x = \int_{\theta 0}^{\theta 1} ds_w - \rho \cos(V)$$

Pedal : the formula to obtain the parametric polar equation of the pedal of the polar curve (ρ_0, θ_0) of a curve given in polar coordinates $(\rho_1, \theta_1$ is :

$$\rho_1 = \rho_0 \sin(V) \text{ and } \theta_1 = \theta_0 - (V - \pi/2)$$

The equation of n^{th} pedal is $(n \in \mathbb{Z})$:

$$\rho_n = \rho_0 \sin^n(V) \text{ and } \theta_n = \theta_0 - n(V - \pi/2)$$

Gregory : The formulas associated to direct (GT) and inverse (GT^{-1}) Gregory's transformation from the wheel to the ground, direct Gregory's Transform is :

$$y = \rho \qquad \text{and} \qquad x = \int \rho . d\theta$$

in the opposite way GT^{-1} from the ground to the wheel :

$$\rho = y \quad \text{and} \quad \theta = \int \frac{dx}{y}$$

$$\tan V = \frac{\rho.d\theta}{d\rho} = \frac{dx}{dy}$$

The GT^{-1} is defined in the whole euclidean plane except on the line $y = 0$ for which the angle of the wheel is not defined. This line is called the "base-line"and is the dual of the pole. A translation in the ground-plane corresponds to a rotation in the wheel-plane.

This article is the 4th part on a total of 12 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.
Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.
Part III : A generalization of sinusoidal spirals and Ribaucour curves
Part IV: Tschirnhausen's cubic.
Part V : Closed wheels and periodic grounds
Part VI : Catalan's curve.
Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, $\beta-$curves.
Part VIII : Translations, rotations, orthogonal trjectories, differential equations, Gregory's transformation.
Part IX : Curves of Duporcq - Sturmian spirals.
Part X : Intrinsically defined plane curves, closed or periodic curves and Gregory's transformation.
Part XI : Inversion, Laguerre T.S.D.R. - Polar tangential and Axial coordinates.
Part XII : Caustics by reflection, curves of direction, rational arc length.

Two papers in french :
1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)
2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988). There is an english adaptation.
Gregory's transformation on the Web : http://christophe.masurel.free.fr

References :

(1) H. Brocard , T. Lemoine - Courbes geometriques remarquables Blanchard Paris 1967 (3 tomes)
(2) F. Gomez Teixeira - Traite des courbes speciales remarquables Chelsea New York 1971 (3 tomes)
(3) Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica
(4) Nouvelles Annales de Mathematiques (1842-1927) NAM (Archives Gallica online)

(5) E. Cesaro - Sur une classe de courbes planes remarquables (NAM 1902).
(6) E. Duporcq NAM 1902 p 181,
(7) A. Mannheim : NAM 1902 p337 et p481,
(8) F. Balitrand : NAM 1914 p97 and M Egan : NAM 1919 p241-248.
(9) H. Wieleitner - Spezielle ebene Kurven p 390 - 1908 Leipzig (G. J. Goschen'sche Verlagshandlung)