

TSCHIRNHAUSEN'S CUBIC

- Part IV -

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Abstract

We present some properties of the Tschirnhausen's Cubic related to the couples ground-wheel and gregory's transformation.

1 Definition :

Tschirnhausen's cubic (TC) has been presented in 1690 in Acta Eruditorum by Walter Tschirnhausen as the caustic by reflection of the parabola for light rays perpendicular to the axis of symmetry. Another definition of (TC) is the antipedal of the parabola w.r.t. its focus.

It is also the third pedal of the point in polar coordinates :

- Line : $\rho = \frac{1}{\cos(\theta)}$ ← first anti-pedal of a point.

- Parabola : $\rho = \frac{1}{\cos^2(\theta/2)}$ ← second anti-pedal

- Tschirnhausen's cubic: $\rho = \frac{1}{\cos^3(\theta/3)}$ ← third anti-pedal

These are three sinusoidal spirals $\rho = \cos^n(\theta/n)$ with $n \in \mathbb{Z}$

Parametric equations (x, y) -plane of this cubic are :

$$x = t - t^3/3 \quad y = t^2$$

2 Cartesian and polar equations of the TC

We need the equation of the TC with vertical axis in an orthonormal frame in the following form :

$$x = t - t^3/3 \quad y = 1 + t^2$$

and :

$$\tan(V) = \frac{dx}{dy} = \frac{1 - t^2}{2t}$$

The coordinates of the focus F is $(0, 4/3)$, the summit S $(0, 1)$ and the double point D $(0, 4)$. The parameter $t = \tan u$ implies $V = \pi/2 - 2u$. The polar equation with the pole at the focus of the TC is:

$$\rho = \frac{1}{3 \cos^3(\theta/3)} \quad \text{with} \quad t = \tan(\theta/3)$$

Three horizontal lines in the plane of TC play a particular role in the geometry of the curve: The directrix/natural base-line ($y=0$), the tangent at S ($y=+1$) and the double normal ($y=+2$).

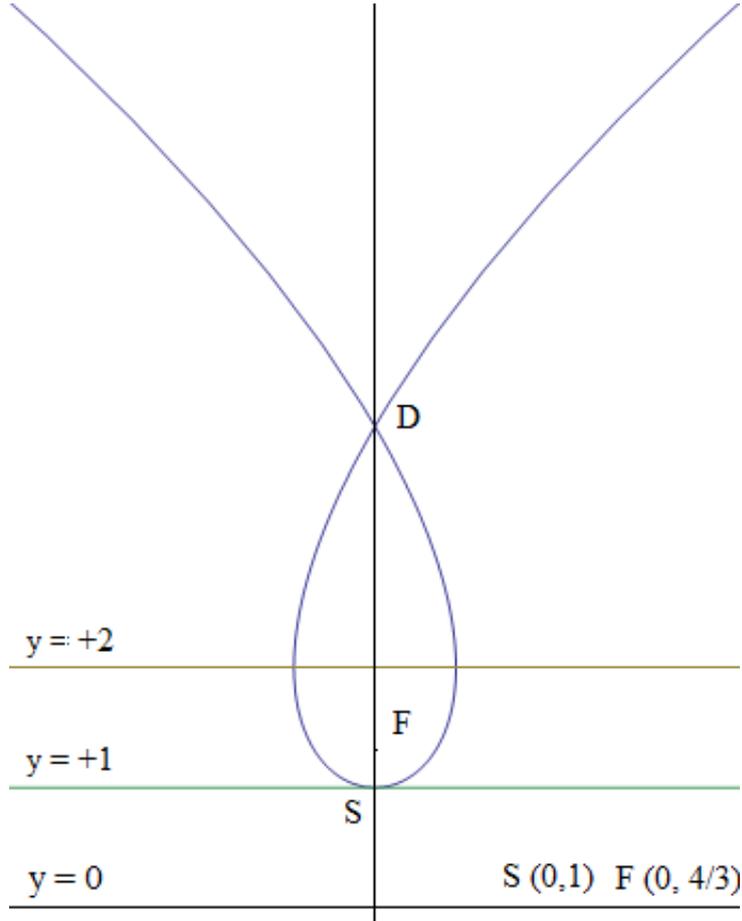


Figure 1: Tschirnhausen's Cubic

3 Equation w.r.t. the focus and the directrix

The $TC(\rho, y)$ can be expressed in the frame : distance ρ to the focus and y to the directrix the same used for conics. With the choice we have made in the equation of the section 2. The directrix is the axis $x'x$. We have seen above that $u = \theta/3$

$$1 + t^2 = \frac{1}{\cos^2(u)} = y$$

$$\rho = \frac{1}{3 \cos^3(\theta/3)}$$

So the equation focus-directrix is : $9\rho^2 = y^3$

4 The TC as a pursuit curve

The pursuit curves for a mobile running along a line are caustics by reflection of parabolas/hyperbolas $y = x^k$ for light rays parallel to $x'x$ axis. For the general case the equation of the pursuit curve is :

$$x = \frac{1}{2} \left[\frac{1}{m+1} t^{m+1} + \frac{1}{m-1} (t^{1-m}) \right] \quad y = t^{1/m}$$

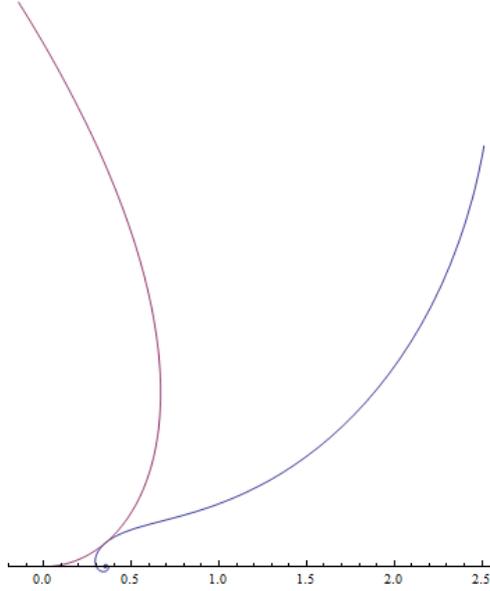


Figure 2: Curve $[\rho = t^2 \quad \theta = t + 1/t]$ rolling on a half-TC base line is tangent at S

For the caustic by reflection with light rays parallele to xx' : $m = k-1$. The parametric equations above define a pursuit curve for a speed of the prosecutor $1/m$ times the speed of the mobile on the line. For the case of the parabola, the pursuit curve corresponds to $m=1/2$. The speed of the pursuiter is twice the speed of the mobile that runs on the tangent line at the summit S is the TC. So the first one meets the mobile at the point S of contact of the tangent with the TC then the pursuit curve is the TC :

$$x = t - t^3/3 \text{ and } y = t^2$$

The wheel associated to this tangent as the base-line is a special case of anallagmatic spirals and is given in parametric polar coordinates by :

$$\rho = t^2 \text{ and } \theta = t + \frac{1}{t}$$

5 Properties derived from the system ground/wheel

In this section we use two base-lines $y=0$ (the directrix) and $y=2$ (the double normal) and the inverse Gregory's transformation (GT^{-1})

We search for the wheel in parametric polar coordinates and for the first one ($y=0$) we find :

$$\rho = y = 1 + t^2 = \frac{1}{\cos^2(u)} \quad \text{and} \quad \theta = \tan(u) - 2u$$

This curve is the Sturm or Norwich spiral, an involute of the involute of the circle. For the other base-line ($y=+2$) and the same TC the wheel is :

$$\rho = t^2 - 1 \quad \text{and} \quad \theta = t \quad \text{so} \quad \rho = \theta^2 - 1$$

These two wheels can roll one on the other around two poles at the distance of 2 with usual conditions. When the Norwich spiral rolls along the xx' axis the pole describes the curve:

$$x = t - t^3/3 \text{ and } y = t^2 - 1$$

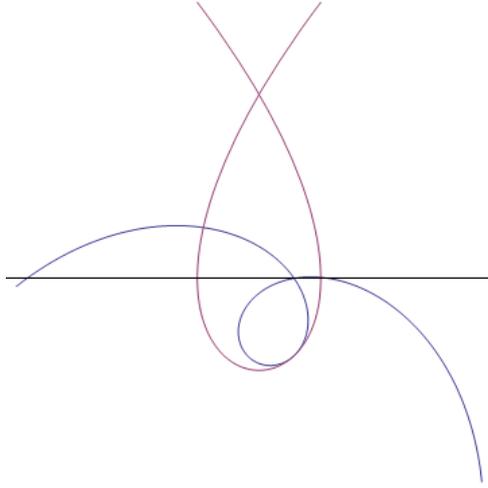


Figure 3: Curve $\rho = \theta^2 - 1$ rolling on a TC base line is double normal

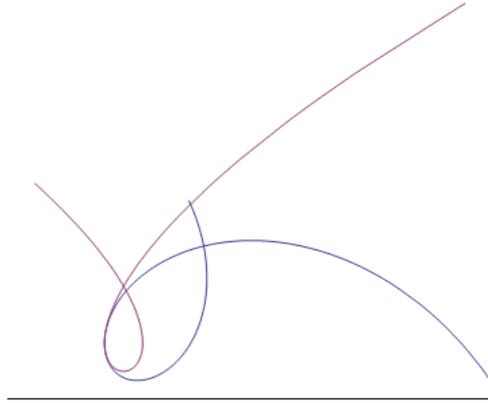


Figure 4: Sturm/Norwich spiral-wheel and TC-ground base line is the directrix

It is a TC for the base-line moved of a length 2 downwards. The pedal of the Norwich Spiral is the curve $\rho = \theta^2 - 1$. So with the theorem 8 (see part (I) Gregory's transformation) : if the Tschirnhausen's Cubic rolls with its linked base-line $y=0$ on a line then the envelope of the base line is the roulette of the pole of the Sturm/Norwich spiral.

It can be shown that this roulette of this spiral is Tschirnhausen's Cubic :

$$x = t - t^3/3 \quad y = t^2 - 1$$

for which the base-line is the double normal. So we have the following result:

Theorem : If a Tschirnhausen's Cubic rolls on a line then the directrix has for envelope an equal TC.

Steiner Habich theorem leads also to this property if we note that the pedal of the Norwich Spiral is $\rho = \theta^2 - 1$ which is a wheel for the TC as the ground and double normal as the base-line.

6 TC as the caustic of the parabola

The property of the TC to be the caustic by reflection of the parabola is proven in any book about this curve for the light rays orthogonal to the axis of the parabola. Knowing the result for this particular direction of the rays it is possible to prove that the caustic is also a TC when the parallel light rays come from any direction except for the one in the direction parallel to the axis of the parabola. In this last case the reflected rays converge to the focus of the parabola. Let's

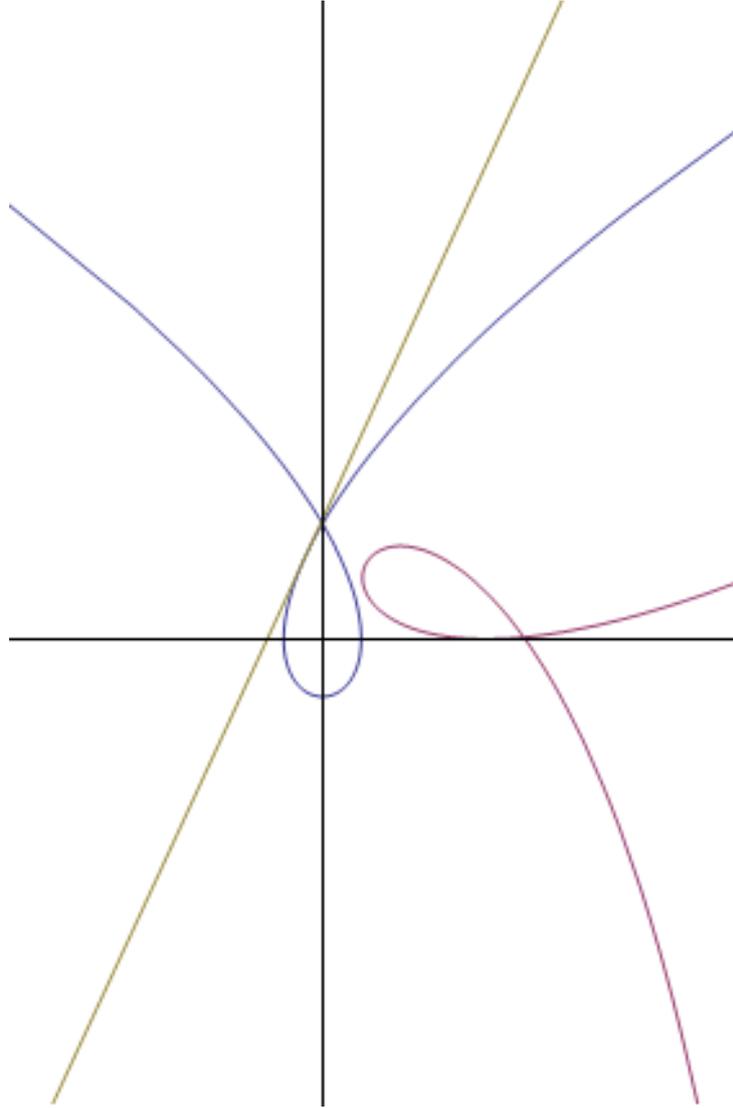


Figure 5: The envelope of linked directrix of TC rolling on a line is an equal TC

suppose the parabola is : $y^2 = 2px$ and the light rays come from ∞ in the direction φ angle with the x 's axis (axis of the Parabola). There is a similarity centered at the common focus of the parabola and the TC of angle $\pi/2 - \varphi$ and ratio $\sin(\varphi)$ which transforms the parabola in the curve described by the projection H of the focus on the reflected ray. It is a parabola and the line HM has for envelope another TC anti-pedal of this last parabola transformed from the initial one by the same similarity of angle $\pi/2 - \varphi$ and ratio $\sin(\varphi)$. The distance from F to the summit S of the TC is $a_\varphi = a_{\pi/2} \cdot \sin(\varphi)$.

7 Curves of Duporcq - Sturmian spirals , Tschirnhausen's cubic and Norwich spiral

In four papers of NAM , E. Duporcq, A. Mannheim (1902), F. Balitrand (1914) and M. Egan (1919) gave the complete solution of the following problem for plane curves : to find the curves such that the element of arc length is e times (e= constant) the segment cut on a given line by the normals at the ends of this arc. If M is the current point of the TC :

$$x_M = t - t^3/3 \quad y_M = t^2 - 1$$

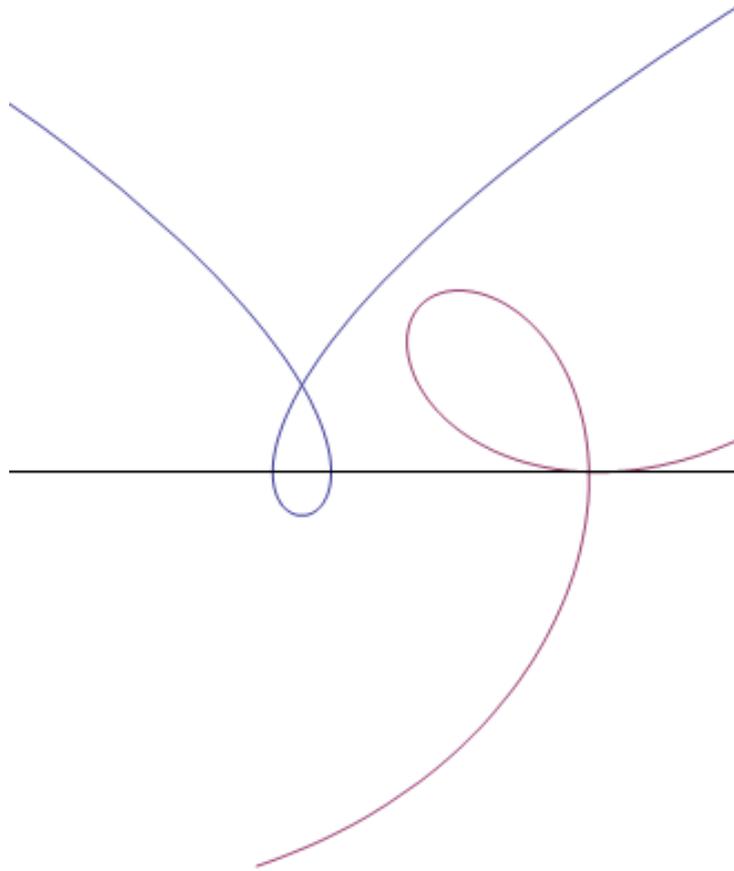


Figure 6: Roulette of Norwich (sturmian) spiral is a Tschirnhausen's cubic (Duporcq curve).

The abscisse of N on the normal of the TC at M for $y=0$ is :

$$x_N = t + t^3/3 = \text{the arc length of TC between } t=0 \text{ and } t=t.$$

So $s(A, B) = \left[t + t^3/3 \right]_{t_0}^{t_1}$ is the arc length between two points A and B on the TC. This result confirms that TC is the solution of Duporcq's problem when $e=1$: equality of the arc and the segment cut by the current normal on the double normal $y=0$ (fig.6/7).

7.1 Duporcq and Bouguer

The sturmian spirals are the curves such that $R_c = e\rho$: ray of curvature equal to e times the vector ray in polar coordinates. Mannheim has shown that the roulette of the pole of a Sturmian spiral is a Duporcq curve.

The special case $e=1$ is the sturmian spiral of Norwich the pole of which describes a Duporcq curve : a Tschirnhausen's cubic when rolling on the double normal of the TC as base line.

The analog problem - but for the tangent - (the arc of the curve is k times, $k=$ speed ratio, the segment cut by the tangents at the ends of the arc on a given line) for a plane curve are the Bouguer curves of pursuit (see Part VII).

And so the TC is a Bouguer pursuit curve for the tangent at the summit with speed ratio $k=2$ and a curve of Duporcq for the double normal for $e=1$ (fig. 7). The couple TC - spiral of Norwich is a special case among the solutions given by M. Egan (NAM 1919) - see part IX -.

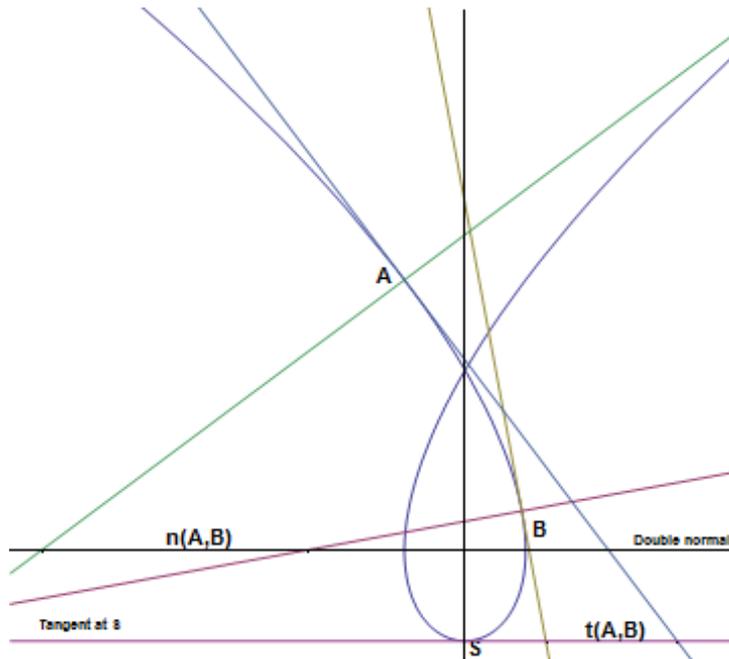


Figure 7: Tschirnhausen's cubic as Duporcq curve and as Pursuit curve. Normals and Tangents at A and B : $s(A, B) = n(A, B) = 2.t(A, B)$

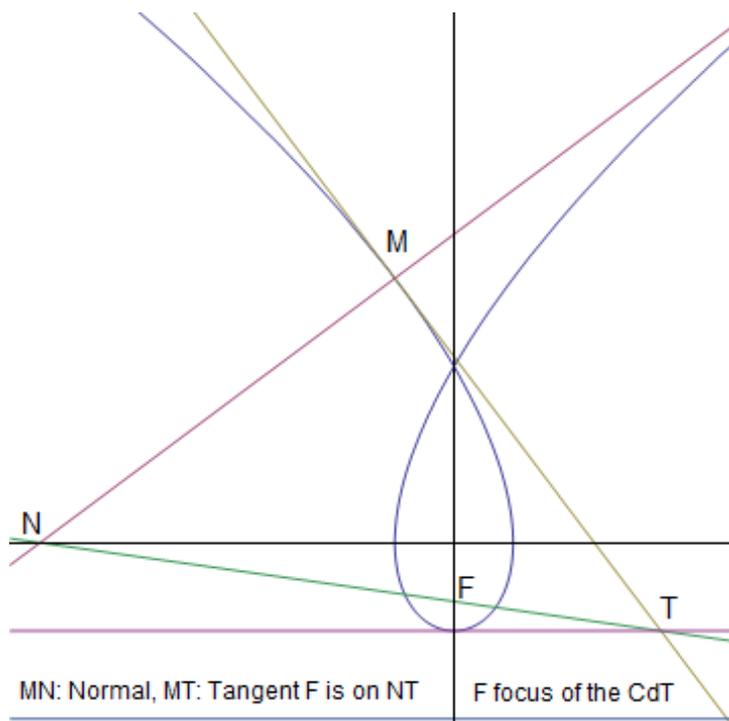


Figure 8: A property of normal and Tangent at M

7.2 A property of the line NT

The normal at M to Tschirnhausen's cubic cuts the double normal in N and the tangent at M intersects the tangent at S in T. Since $x_N = -2.x_T$ the focus $F(-2/3, 0)$ is on the line NT (fig.8).

8 A property Linking the Parabola and its directrix and the TC (Steve Phelps)

There is a simple link between the parabola $x = 2t, y = 1 + t^2$ with x-axis as the directrix and Tschirnhausen's cubic. Consider the current normal at M on the parabola. We call N the point of intersection of this normal with the directrix $x'x$. Inside the parabola on extended NM we take a length $\overline{MP} = 2\overline{NM}$. Then the point P will run on a Tschirnhausen's cubic with same focus.

Parametric equations (x, y) of this TC are :

$$x = \frac{1}{2} \cdot (3t - t^3) \quad y = \frac{3}{2} \cdot (1 + t^2)$$

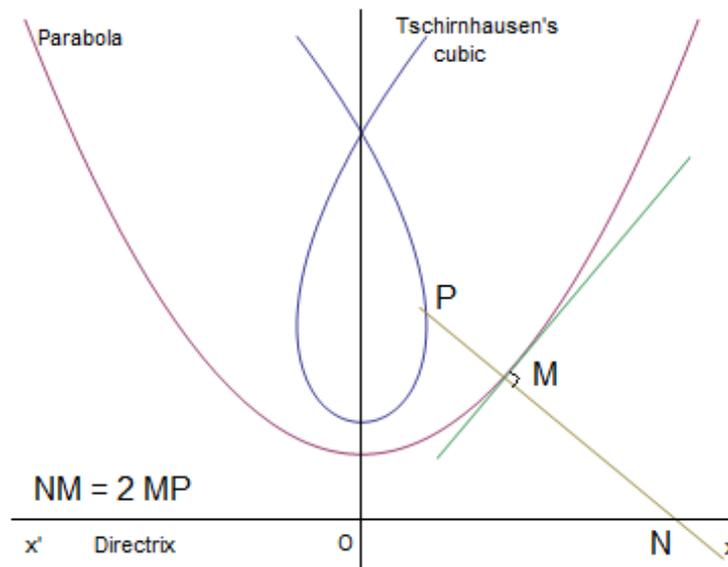


Figure 9: A geometric link between Parabola and TC

9 A property of the osculating circle of the TC

The symmetric of the osculating circle at $M \in TC$ w.r.t. to the tangent at the same point M is tangent to the directrix.

Annexe I :

Roulette, Pedal and Gregory's transformations.

Parametric equations of the roulette (x, y) of the pedal (ρ, θ) and Gregory's transformations (GT, GT^{-1})

Roulette : the formulas to determine the cartesian parametric equation (x, y) of the roulette on the line xx' of a curve given in polar coordinates (ρ, θ) :

$$y = \rho \sin(\theta) \text{ and } x = \int_{\theta_0}^{\theta_1} ds_w - \rho \cos(V)$$

Pedal : the formula to obtain the parametric polar equation of the pedal of the polar curve (ρ_0, θ_0) of a curve given in polar coordinates (ρ_1, θ_1) is :

$$\rho_1 = \rho_0 \sin(V) \text{ and } \theta_1 = \theta_0 - (V - \pi/2)$$

The equation of n^{th} pedal is ($n \in \mathbb{Z}$) :

$$\rho_n = \rho_0 \sin^n(V) \text{ and } \theta_n = \theta_0 - n(V - \pi/2)$$

Gregory : The formulas associated to direct (GT) and inverse (GT^{-1}) Gregory's transformation from the wheel to the ground, direct Gregory's Transform is :

$$y = \rho \quad \text{and} \quad x = \int \rho \cdot d\theta$$

in the opposite way GT^{-1} from the ground to the wheel :

$$\rho = y \quad \text{and} \quad \theta = \int \frac{dx}{y}$$
$$\tan V = \frac{\rho \cdot d\theta}{d\rho} = \frac{dx}{dy}$$

The GT^{-1} is defined in the whole euclidean plane except on the line $y = 0$ for which the angle of the wheel is not defined. This line is called the "base-line" and is the dual of the pole. A translation in the ground-plane corresponds to a rotation in the wheel-plane.

This article is the IV^{th} part on a total of 8 papers on Gregory's transformation and related topics.

Part I : Gregory's transformation.

Part II : Gregory's transformation Euler/Serret curves with same arc length as the circle.

Part III : A generalization of sinusoidal spirals and Ribaucour curves

Part IV: Tschirnhausen's cubic.

Part V : Closed wheels and periodic grounds

Part VI : Catalan's curve.

Part VII : Anallagmatic spirals, Pursuit curves, Hyperbolic-Tangentoid spirals, β -curves.

Part VIII : Translations, rotations, orthogonal trajectories, differential equations, Gregory's transformation.

Part IX : Curves of Duporcq - Sturmian spirals.

Part X : Intrinsically defined plane curves, closed or periodic curves and Gregory's transformation.

Two papers in french :

1- Quand la roue ne tourne plus rond - Bulletin de l'IREM de Lille (no 15 Fevrier 1983)

2- Une generalisation de la roue - Bulletin de l'APMEP (no 364 juin 1988).

There is an english adaptation.

Gregory's transformation on the Web : <http://christophe.masurel.free.fr>

References :

H. Brocard , T. Lemoine - Courbes geometriques remarquables Blanchard Paris 1967 (3 tomes)

F. Gomez Teixeira - Traite des courbes speciales remarquables Chelsea New York 1971 (3 tomes)

Journal de mathematiques pures et appliquees (1836-1934) Archives Gallica

Nouvelles Annales de Mathematiques (1842-1927) NAM (Archives Gallica online)

E. Duporcq NAM 1902 p 181,

A. Mannheim : NAM 1902 p337 et p481,

F. Balitrand : NAM 1914 p97 and M Egan : NAM 1919 p241-248.

Geogebra animation : Tschirnhausen cubic <https://www.geogebra.org/m/YXqHDeDS> Author : Steve Phelps.